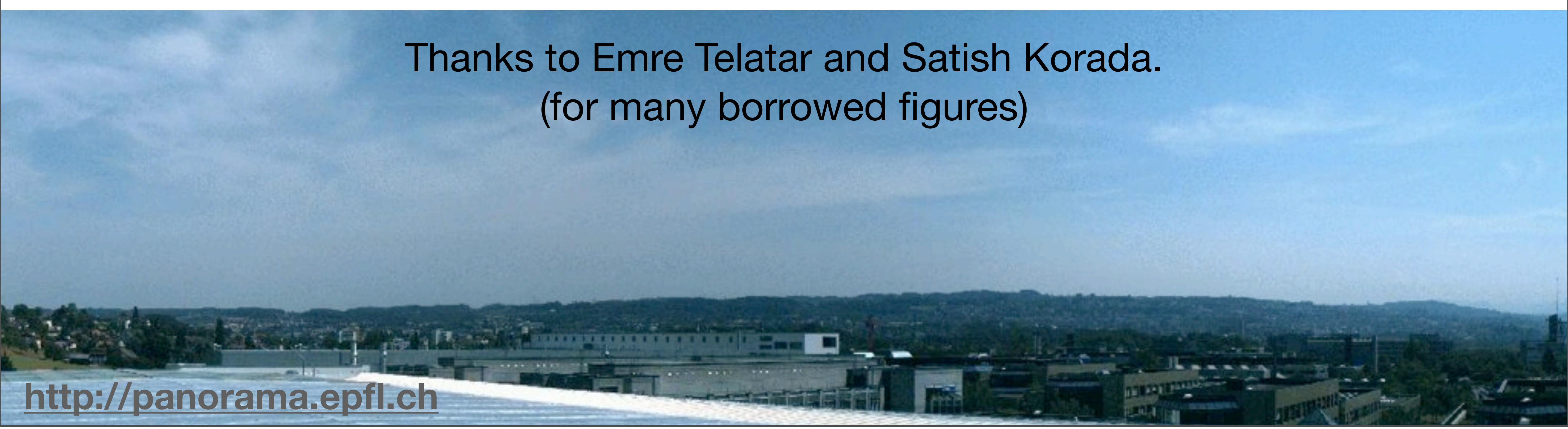
Polar Codes -- A New Paradigm for Coding

R. Urbanke, EPFL

Physics of Algorithms, Santa Fe, September 2nd, 2009



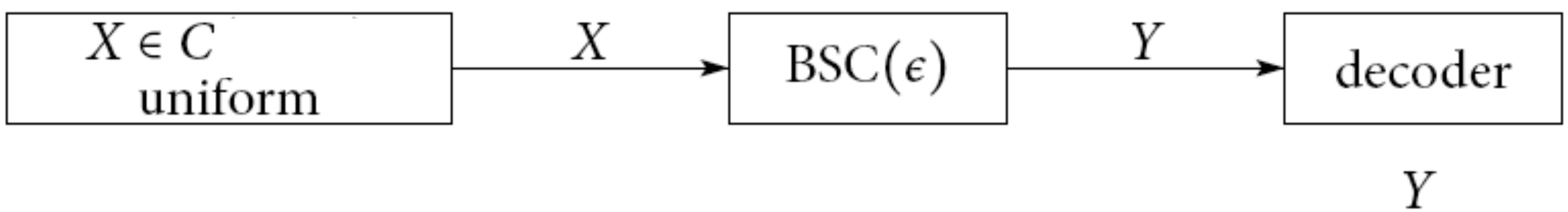
Thanks to Emre Telatar and Satish Korada.
(for many borrowed figures)

http://panorama.epfl.ch

Coding

Coding

Coding



C={000, 010, 101, 111}   code

n ... blocklength

Important Parameters

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$(r, \mathrm{P}, \chi_E, \chi_D, n)$

rate, error probability,
encoding complexity,
decoding complexity,
blocklength

Linear Codes

Linear Codes

$$C(G) = \left\{ x \in \mathbb{F}^n : x = uG, u \in \mathbb{F}^k \right\}$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

generator matrix

Linear Codes

$$C(G) = \{x \in \mathbb{F}^n : x = uG, u \in \mathbb{F}^k\} \qquad C = \{x \in \mathbb{F}^n : x = uG, u \in \mathbb{F}^k\} = \{x \in \mathbb{F}^n : Hx^T = 0^T\}$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

generator matrix

parity-check matrix

Linear Codes

$$C(G) = \left\{ x \in \mathbb{F}^n : x = uG, u \in \mathbb{F}^k \right\} \qquad C = \left\{ x \in \mathbb{F}^n : x = uG, u \in \mathbb{F}^k \right\} = \left\{ x \in \mathbb{F}^n : Hx^T = 0^T \right\}$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

generator matrix

parity-check matrix

Bitwise MAP Decoding

$$
\begin{aligned}
\hat{x}_i^{\mathrm{MAP}}(y) &= \mathrm{argmax}_{x_i \in \{\pm 1\}} p_{X_i \mid Y}(x_i \mid y) \\
\text{(law of total probability)} \quad &= \mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} p_{X \mid Y}(x \mid y) \\
\text{(Bayes')} \quad &= \mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} p_{Y \mid X}(y \mid x) p_X(x) \\
\text{(2.13)} \quad &= \mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} \Big(\prod_j p_{Y_j \mid X_j}(y_j \mid x_j) \Big) \mathbb{1}_{\{x \in C\}},
\end{aligned}
$$

Bitwise MAP Decoding

$$\hat{x}_i^{\mathrm{MAP}}(y) = \mathrm{argmax}_{x_i \in \{\pm 1\}} p_{X_i \mid Y}(x_i \mid y)$$

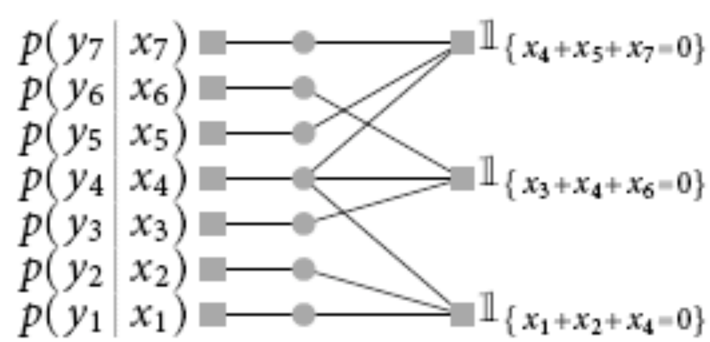
(law of total probability) $$= \mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} p_{X \mid Y}(x \mid y)$$

(Bayes’) $$= \mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} p_{Y \mid X}(y \mid x) p_X(x)$$

(2.13) $$= \mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} \Big(\prod_j p_{Y_j \mid X_j}(y_j \mid x_j) \Big) \mathbb{1}_{\{x \in C\}},$$

$$\mathrm{argmax}_{x_i \in \{\pm 1\}} \sum_{\sim x_i} \Big(\prod_{j=1}^{7} p_{Y_j \mid X_j}(y_j \mid x_j) \Big) \mathbb{1}_{\{x_1+x_2+x_4=0\}} \mathbb{1}_{\{x_3+x_4+x_6=0\}} \mathbb{1}_{\{x_4+x_5+x_7=0\}}$$

$$H = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$



[LDPC -- Gallager ‘60]

Polar Codes: Summary

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Erdal Arikan, ISIT 2007

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very general phenomenon

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information theoretic view why codes work

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first “low complexity” scheme which provably achieves the capacity for a fairly wide array of channels

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many possible variations on the theme

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information theoretic view why codes work

first “low complexity” scheme which provably achieves the capacity for a fairly wide array of channels

many possible variations on the theme

codes not only good for channel coding; work equally well for source coding and more complicated scenarios

References

References

arXiv:0807.3917 [ps, pdf, other]

Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels

Erdal Arikan

Comments: 49 pages, 14 figures. Submitted to IEEE Transactions on Information Theory, Oct. 2007

Subjects: **Information Theory (cs.IT)**

arXiv:0807.3806 [ps, pdf, other]

On the Rate of Channel Polarization

Erdal Arikan, Emre Telatar

Comments: 10 pages

Subjects: **Information Theory (cs.IT)**

arXiv:0901.2370 [ps, pdf, other]

Polar Codes for Channel and Source Coding

Nadine Hussami, Satish Babu Korada, Rudiger Urbanke

Comments: submitted to ISIT

Subjects: **Information Theory (cs.IT)**

arXiv:0901.0536 [ps, pdf, other]

Polar Codes: Characterization of Exponent, Bounds, and Constructions

Satish Babu Korada, Eren Sasoglu, Rudiger Urbanke

Comments: Submitted to IEEE Transactions on Information Theory, minor updates

Subjects: **Information Theory (cs.IT)**

arXiv:0901.2207 [ps, pdf, other]

Performance and Construction of Polar Codes on Symmetric Binary-Input Memoryless Channels

Ryuhei Mori, Toshiyuki Tanaka

Comments: 5 pages, 3 figures, submitted to ISIT2009

Subjects: **Information Theory (cs.IT)**

arXiv:0903.0307 [ps, pdf, other]

Polar Codes are Optimal for Lossy Source Coding

Satish Babu Korada, Rudiger Urbanke

Comments: 15 pages, submitted to Transactions on Information Theory

Subjects: **Information Theory (cs.IT)**

arXiv:0907.3291 [ps, pdf, other]

The Compound Capacity of Polar Codes

S. Hamed Hassani, Satish Babu Korada, Ruediger Urbanke

Comments: 5 pages

Subjects: **Information Theory (cs.IT)**

arXiv:0908.0302 [ps, pdf, other]

Polarization for arbitrary discrete memoryless channels

Eren Sasoglu, Emre Telatar, Erdal Arikan

Comments: 12 pages

Subjects: **Information Theory (cs.IT)**

Codes from Kronecker Product of G_2

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$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Codes from Kronecker Product of G_2

$$G_2^{\otimes 2} = \begin{bmatrix} G_2 & 0 \\ G_2 & G_2 \end{bmatrix}$$

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Codes from Kronecker Product of G$_2$

$$G_2^{\otimes 3} = \begin{bmatrix} G_2 & 0 & 0 & 0 \\ G_2 & G_2 & 0 & 0 \\ G_2 & 0 & G_2 & 0 \\ G_2 & G_2 & G_2 & G_2 \end{bmatrix}$$

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Codes from Kronecker Product of G_2

length $N = 2^m$, $m \in \mathbb{N}$

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generator matrix: rows of $G_2^{\otimes m}$

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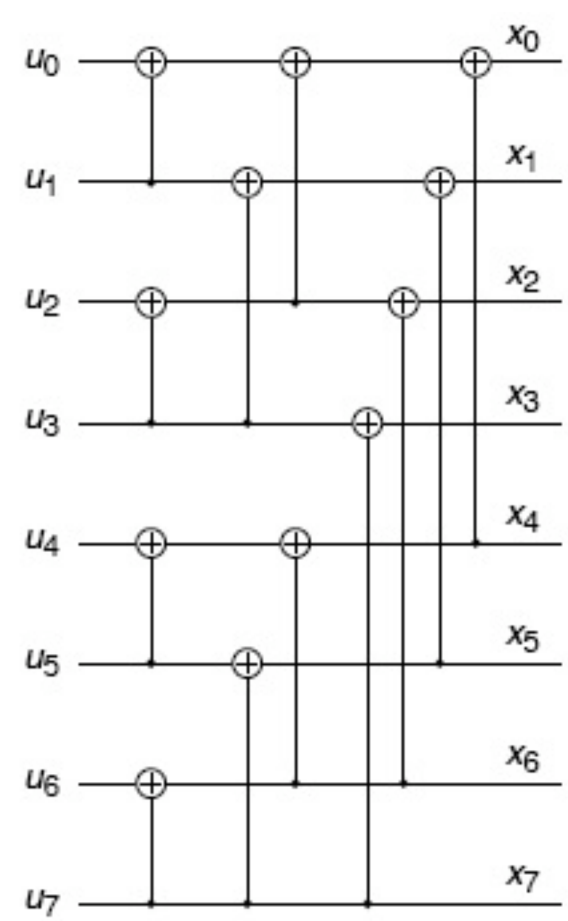

$$\bar{x} = [u_0\ u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7] G_2^{\otimes 3}$$

Codes from Kronecker Product of G_2

length $N = 2^m$, $m \in \mathbb{N}$

generator matrix: rows of $G_2^{\otimes m}$

$$G_2^{\otimes 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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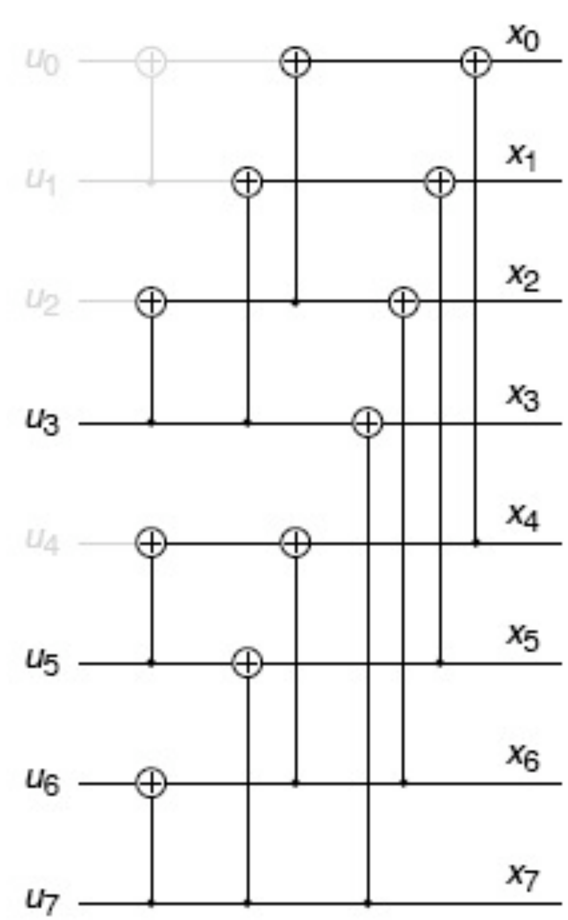

Codes from Kronecker Product of G_2

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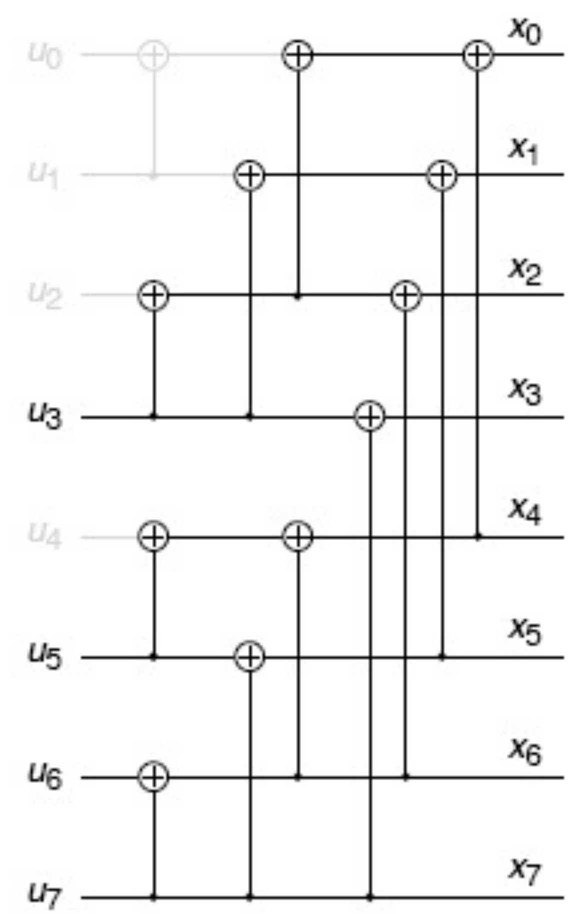


How to choose the rows?

Reed-Muller Codes

length $N = 2^m$, $m \in \mathbb{N}$

generator matrix: rows of $G_2^{\otimes m}$

How to choose the rows?

$$G_2^{\otimes 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\bar{x} = [u_0 \; u_1 \; u_2 \; u_3 \; u_4 \; u_5 \; u_6 \; u_7] G_2^{\otimes 3}$$

Reed-Muller Codes

length $N = 2^m$, $m \in \mathbb{N}$

generator matrix: rows of $G_2^{\otimes m}$

How to choose the rows?

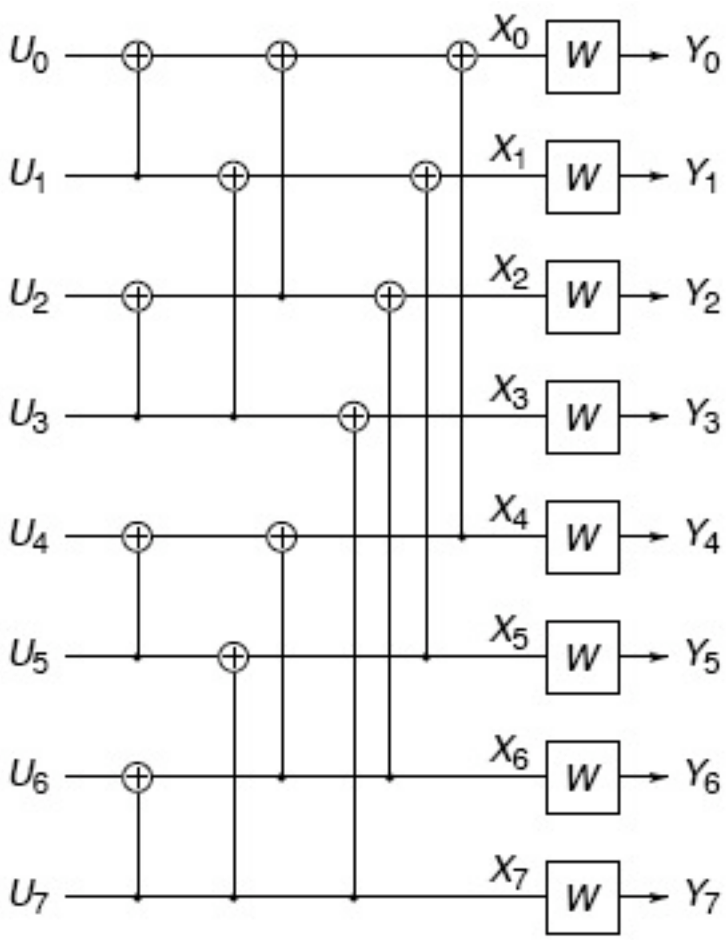
$$G_2^{\otimes 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

choose rows of largest weight

$$\bar{x} = [0\ 0\ 0\ u_3\ 0\ u_5\ u_6\ u_7] G_2^{\otimes 3}$$

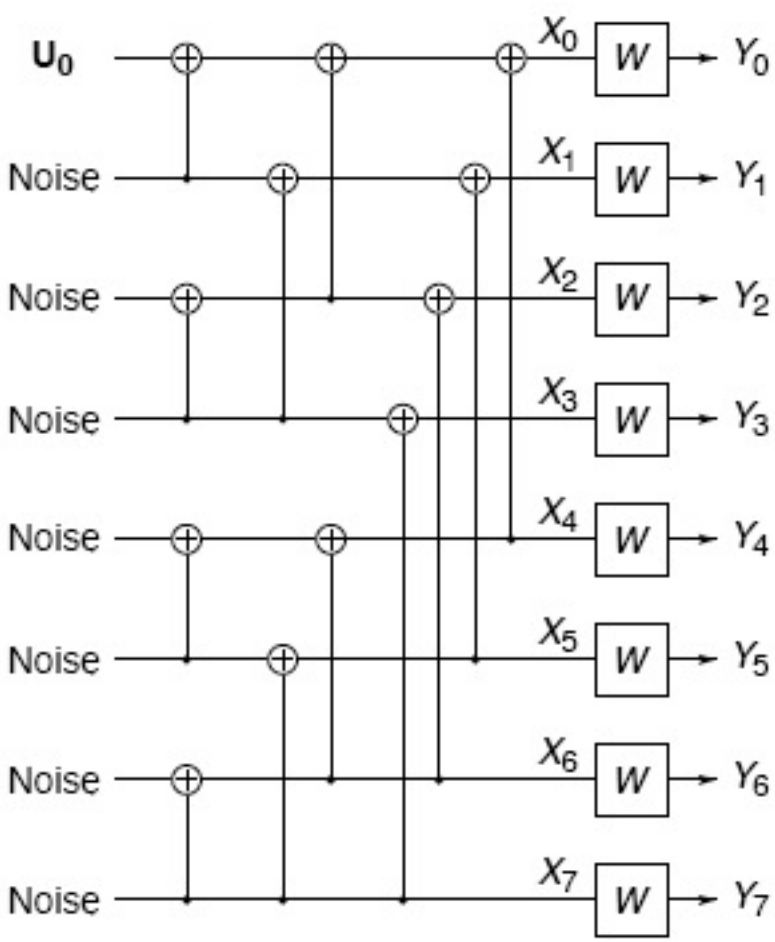
Polar Codes

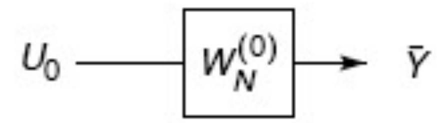
Polar Codes



W -- BMS channel

Polar Codes





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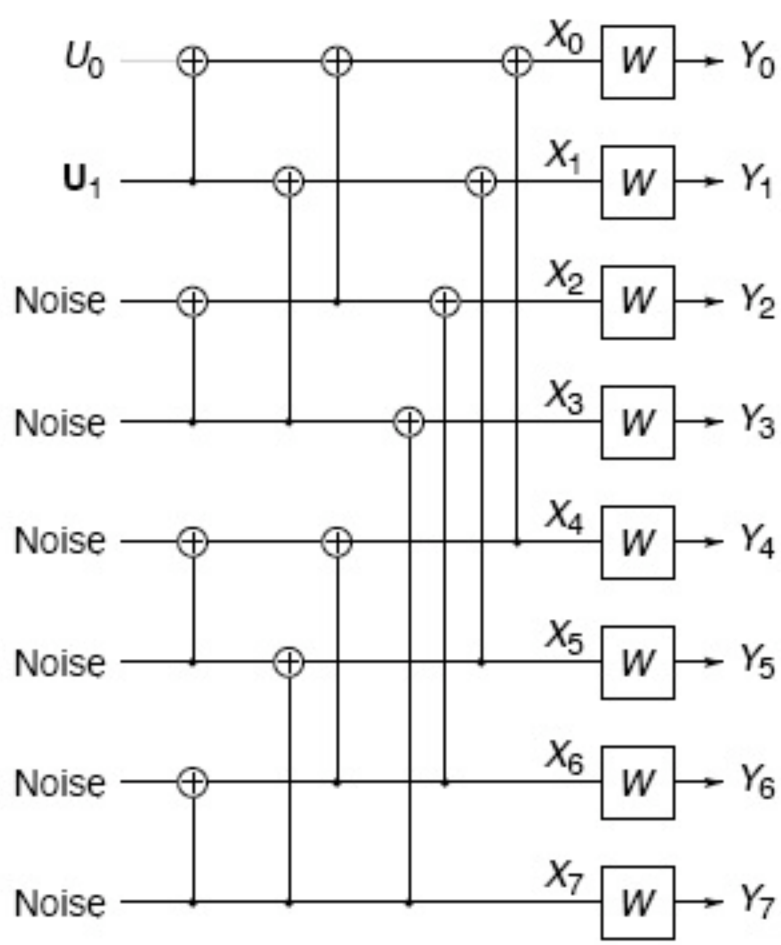

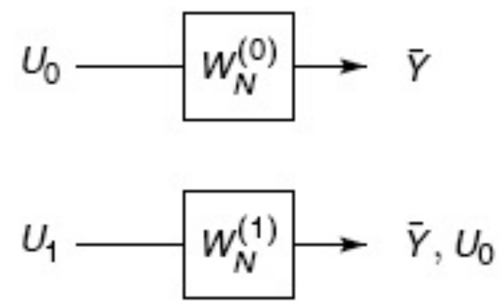

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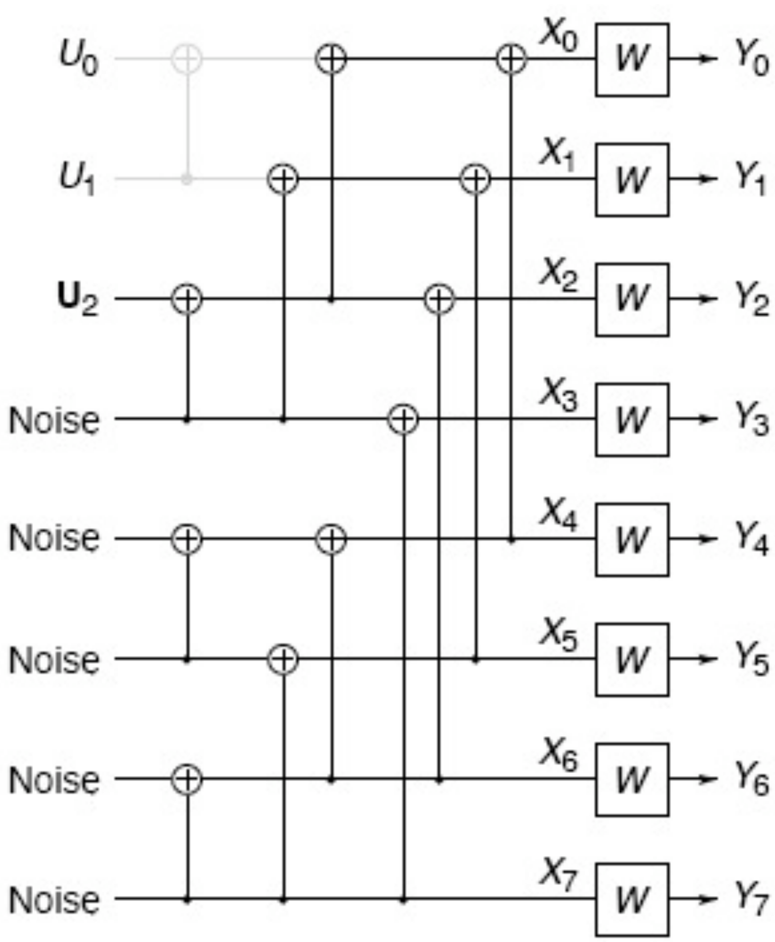

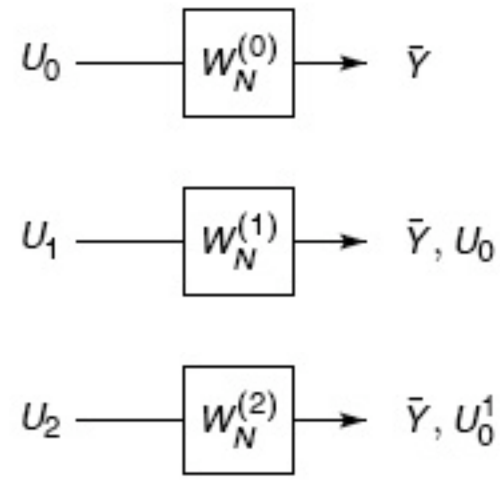

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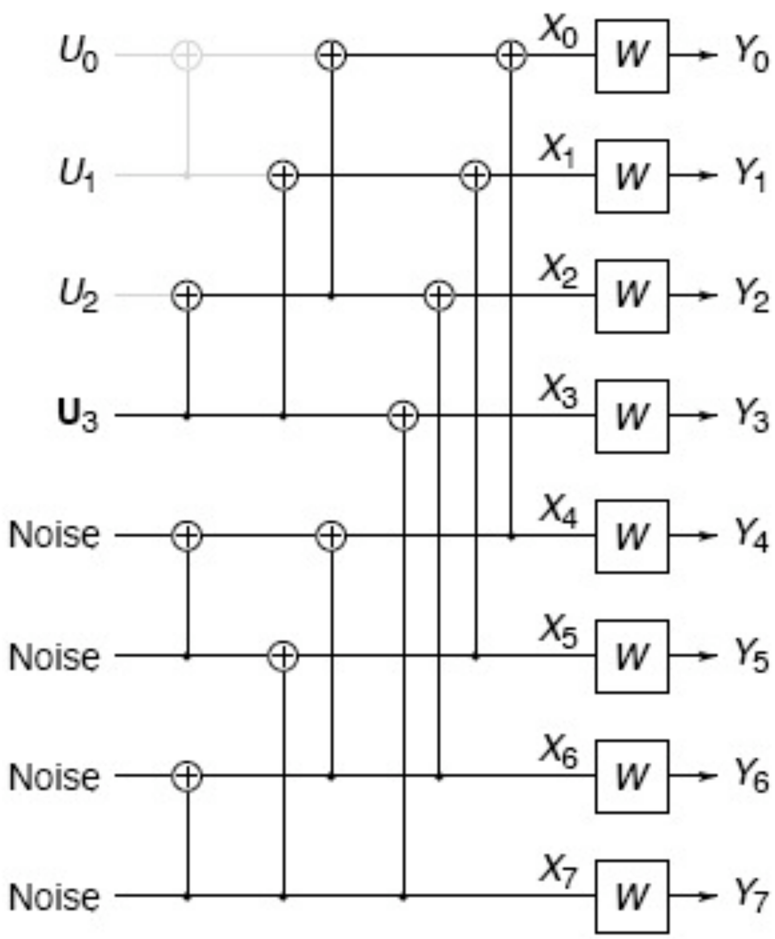
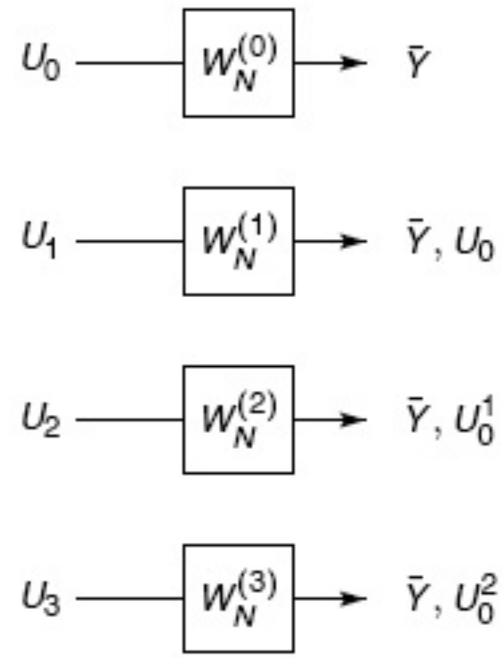
Polar Codes





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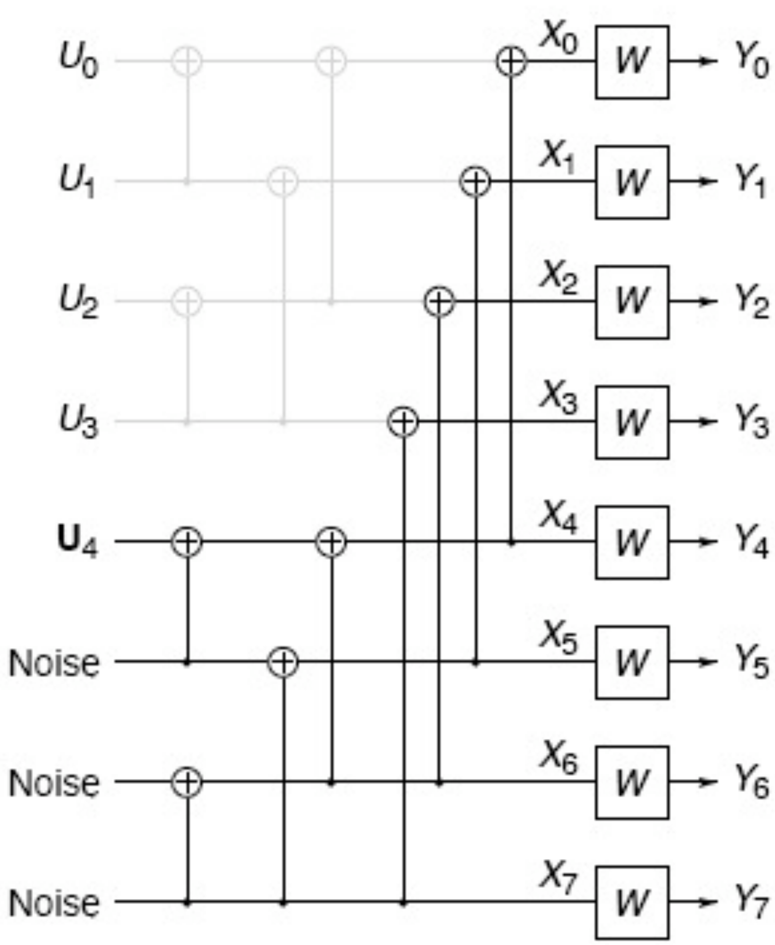
Polar Codes



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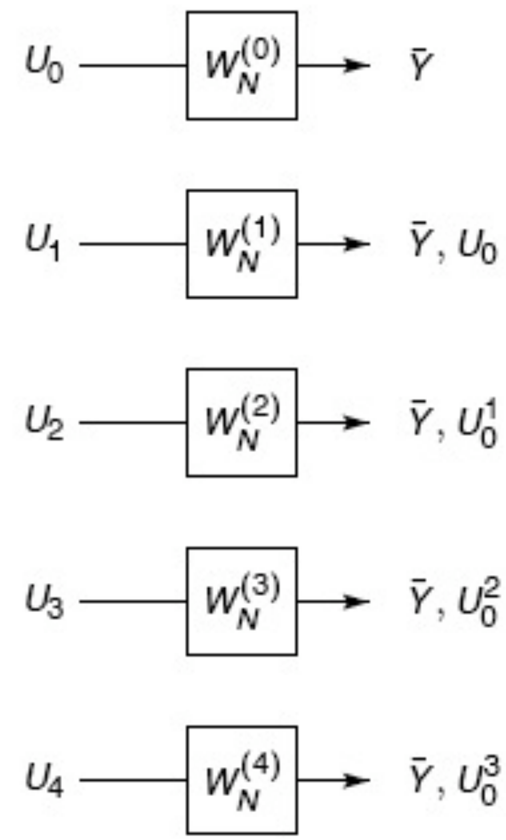
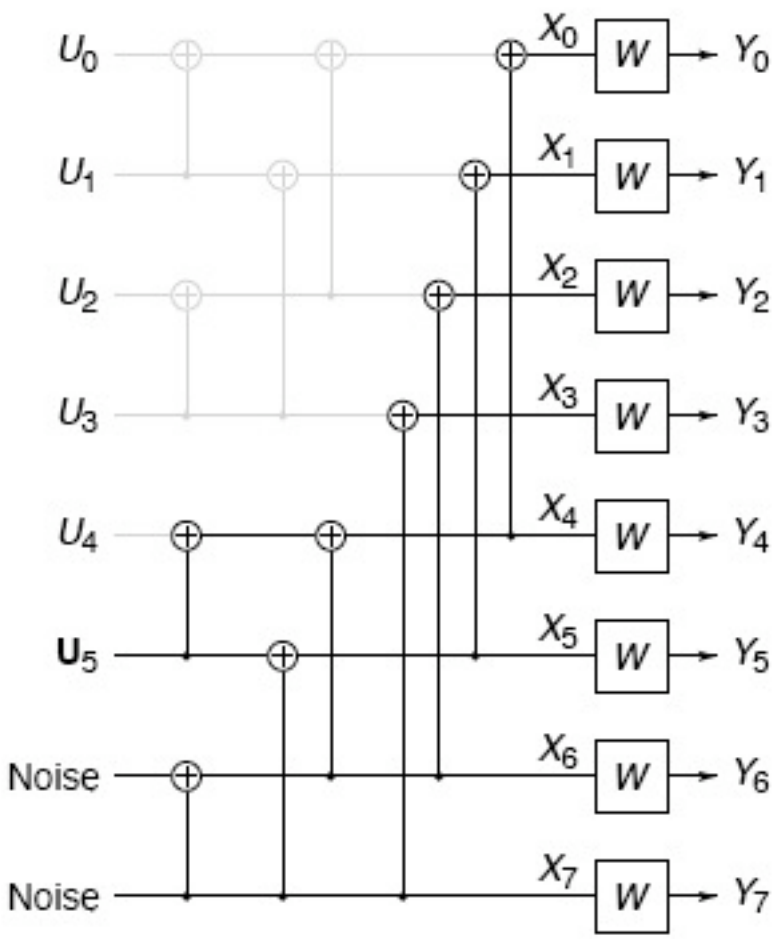
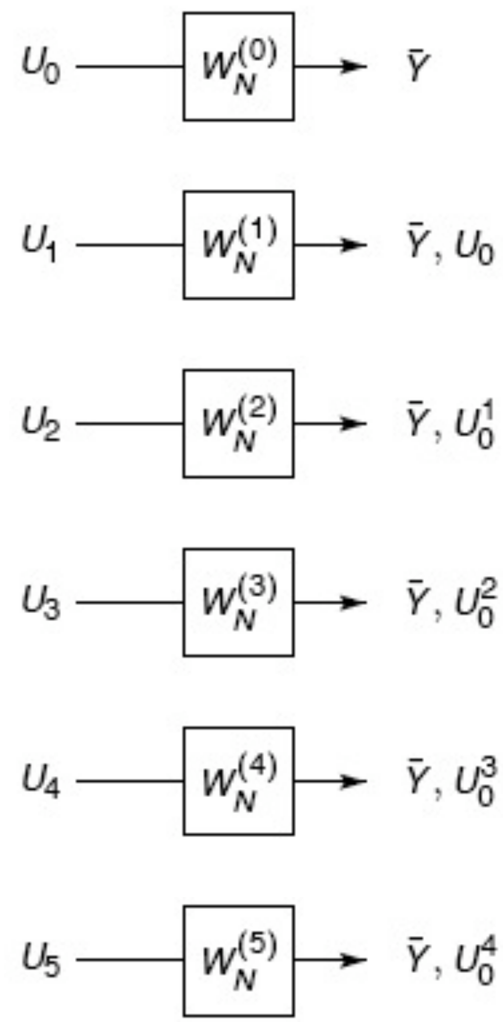


Polar Codes





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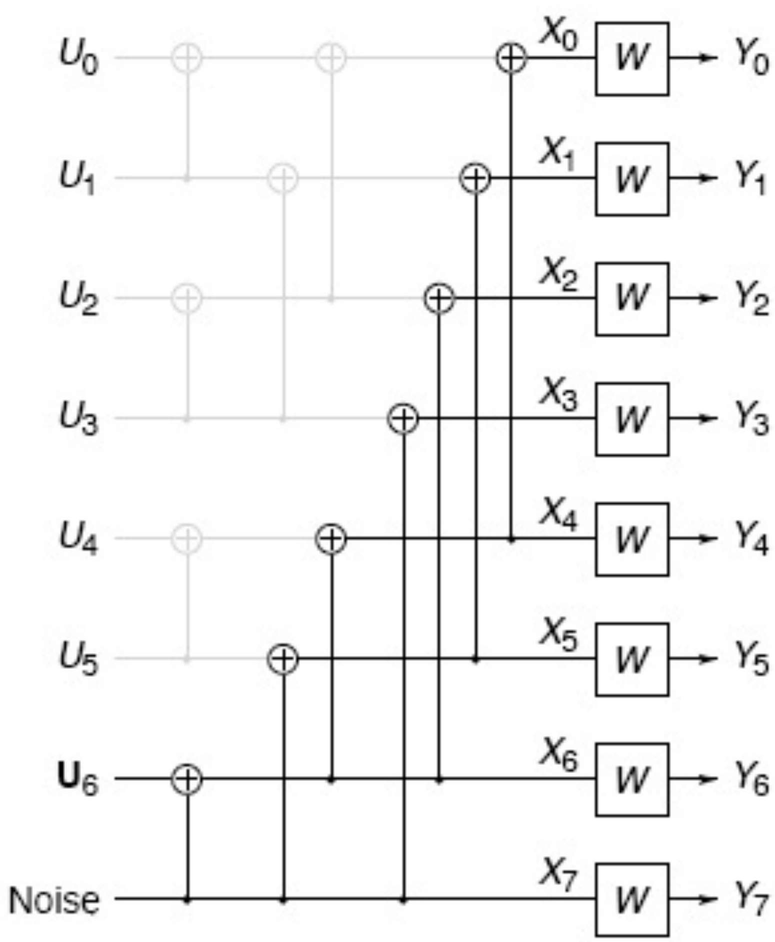

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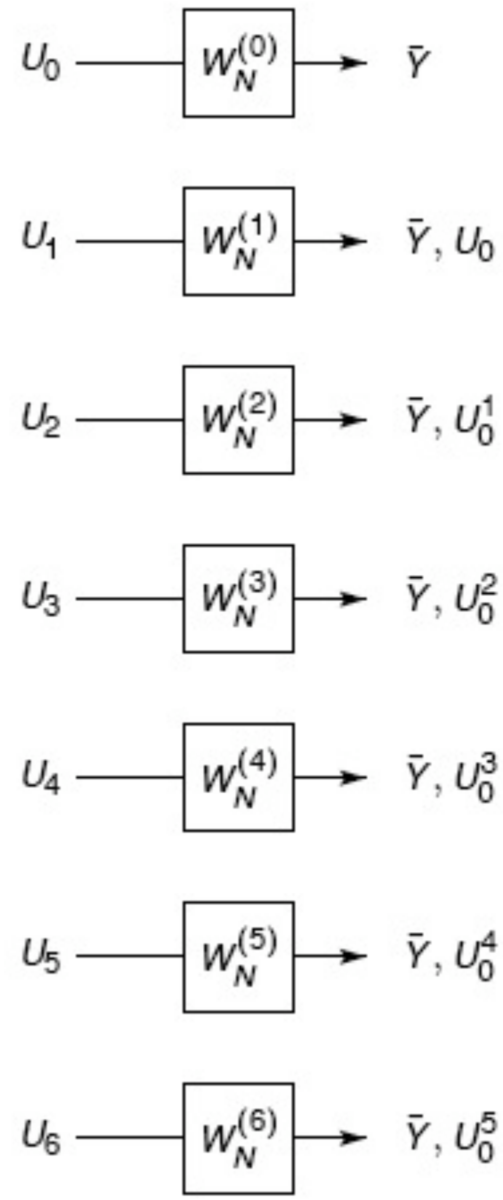

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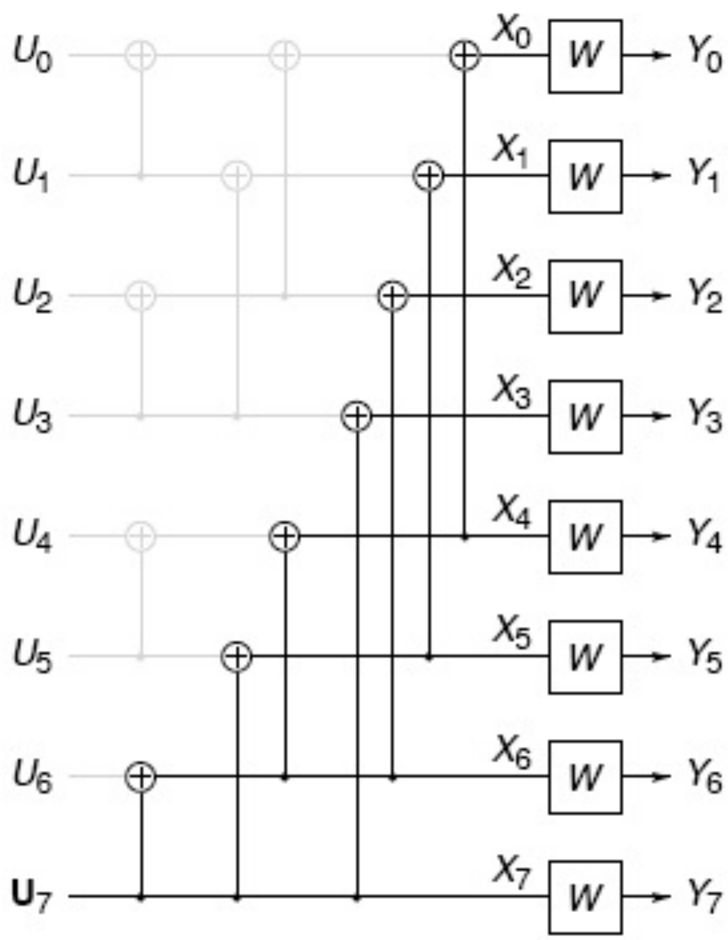

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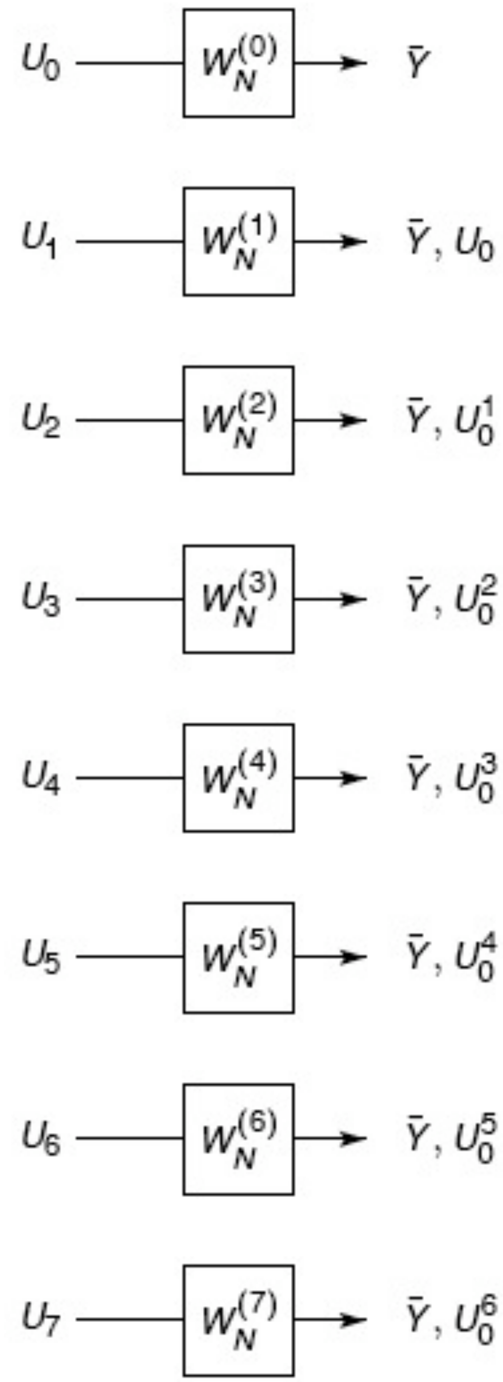
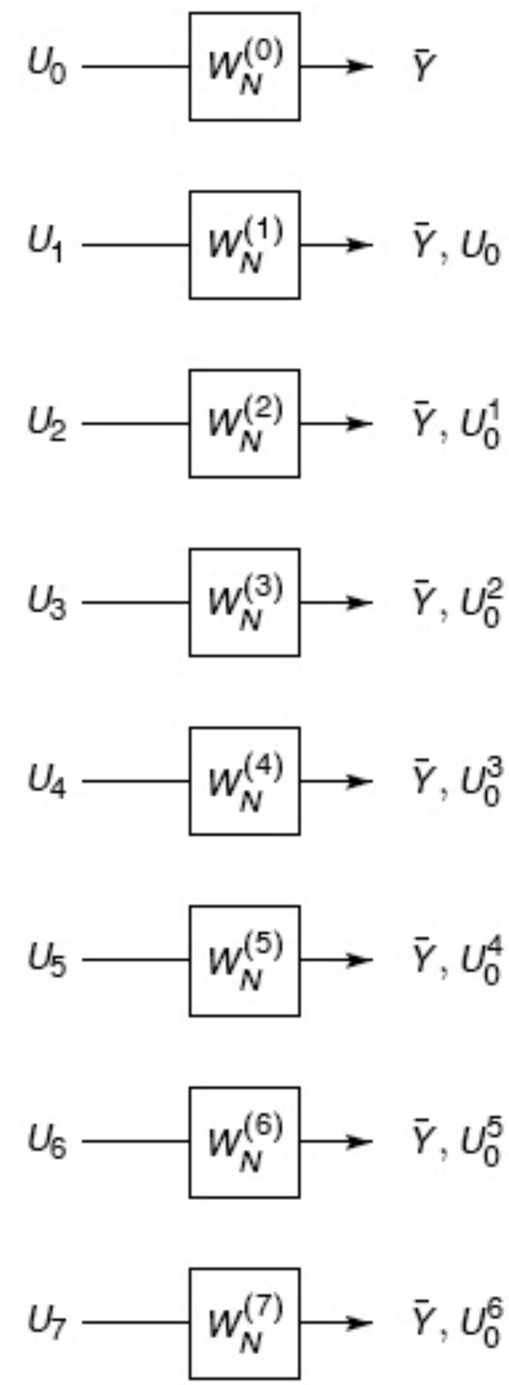
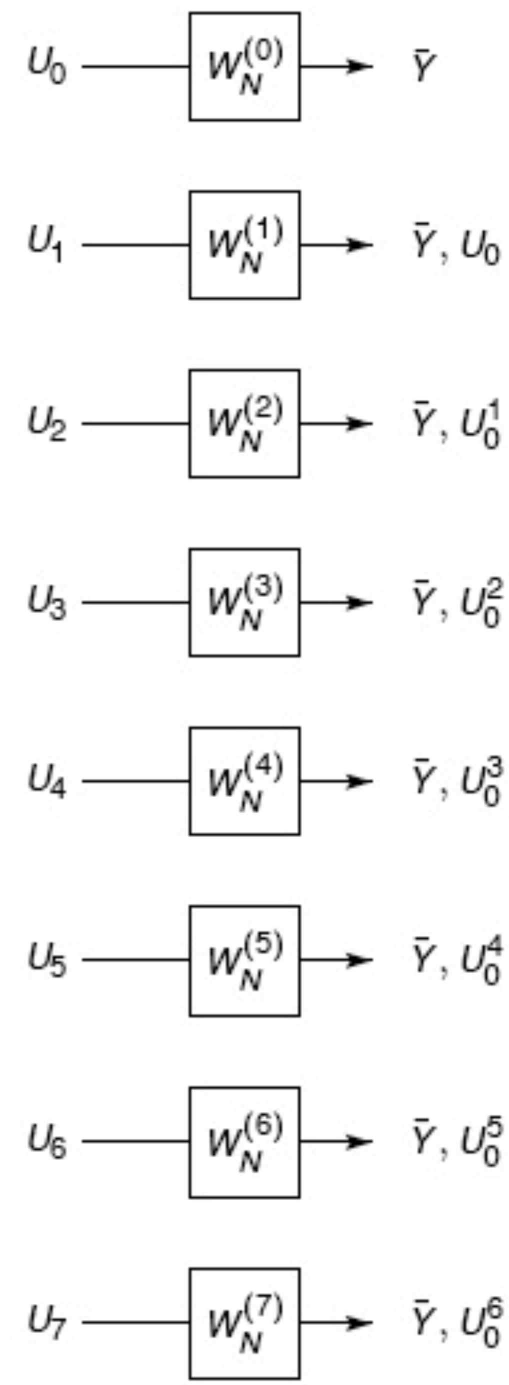


Channel Polarization



Channel Polarization

as $N \to \infty$, channels polarize, either completely noisy or noise-free



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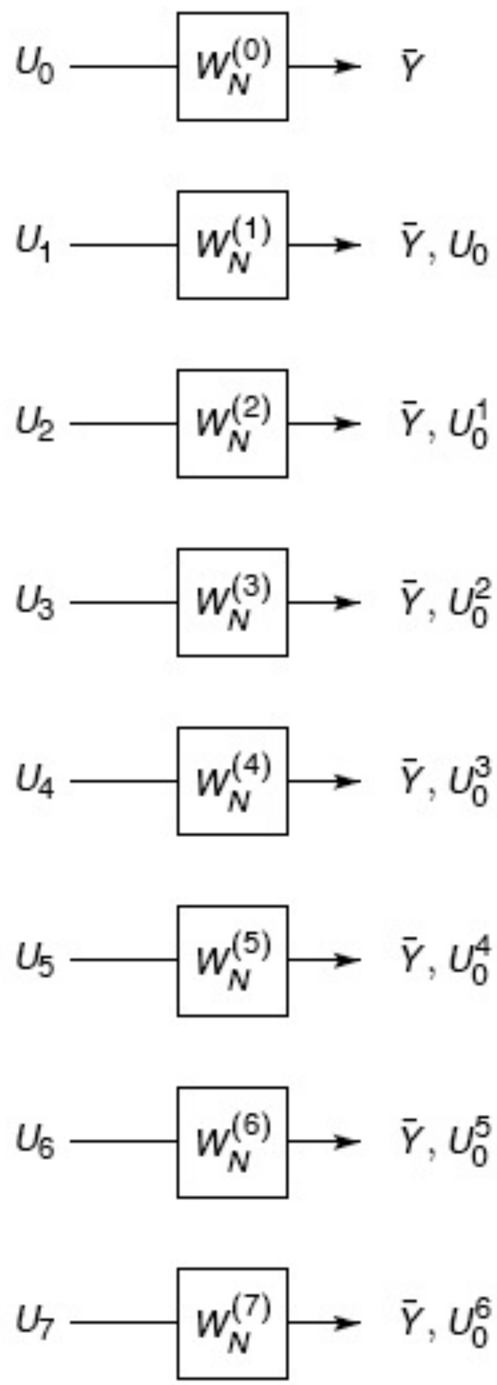
fraction of good channels approaches capacity $I(W)$



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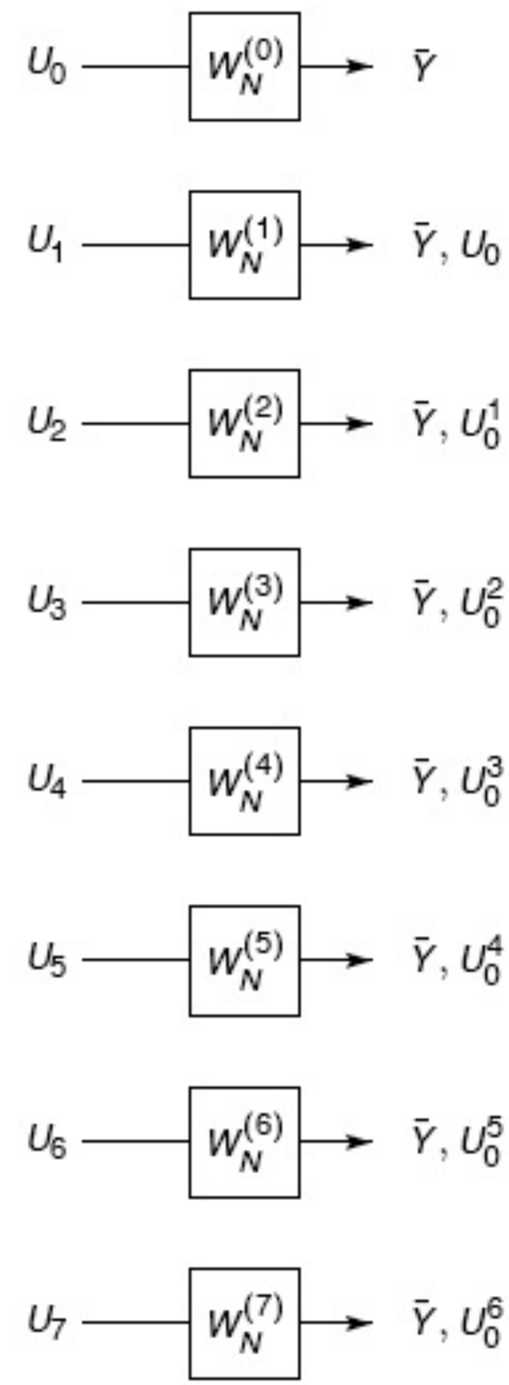
fix bad channels and transmit uncoded bits over the good ones



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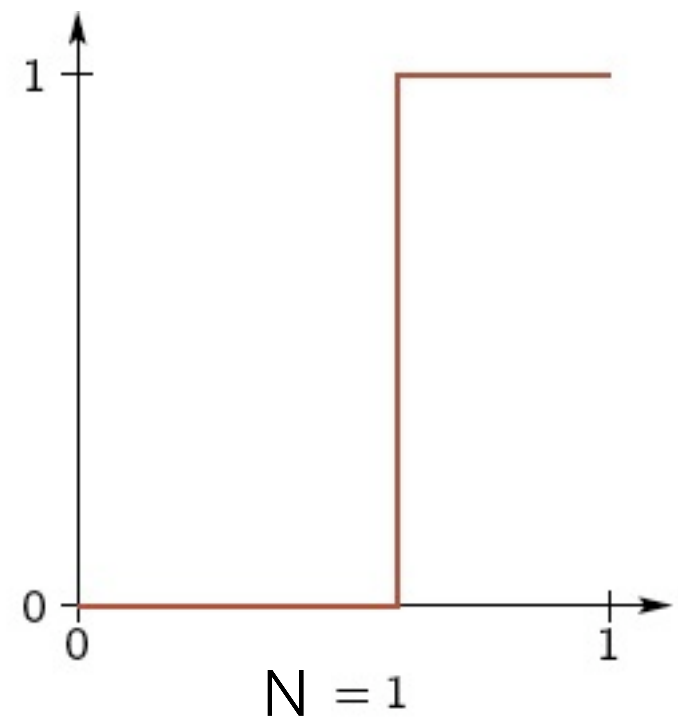
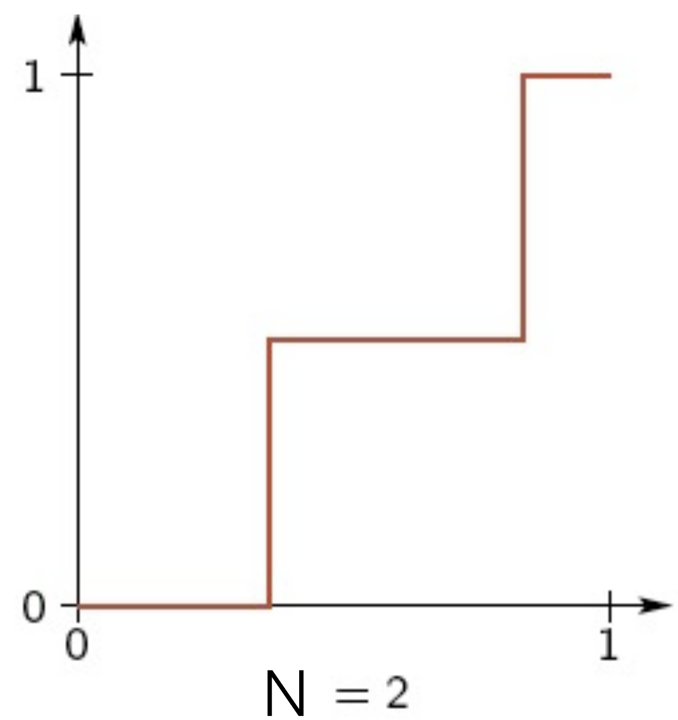


Channel Polarization



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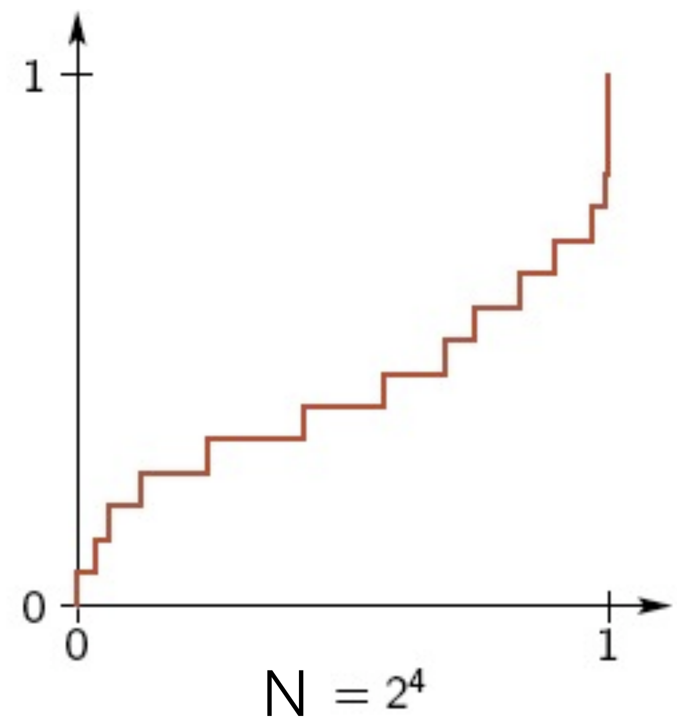

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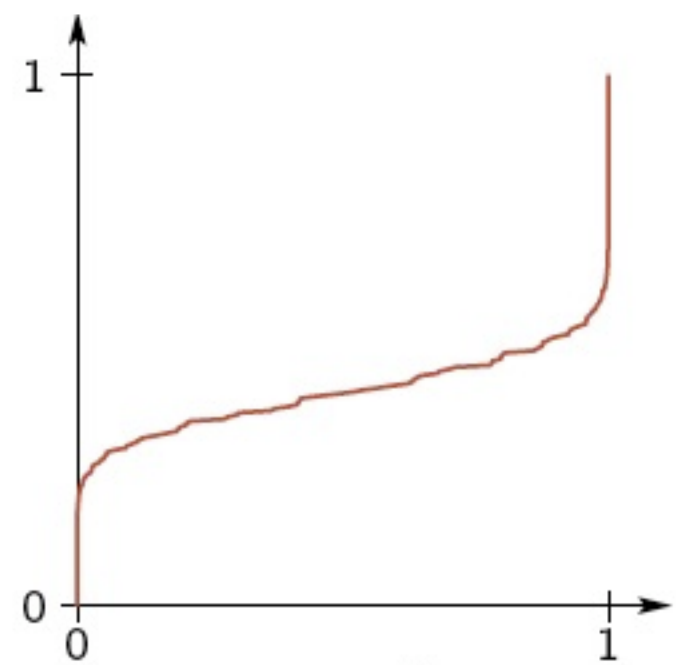
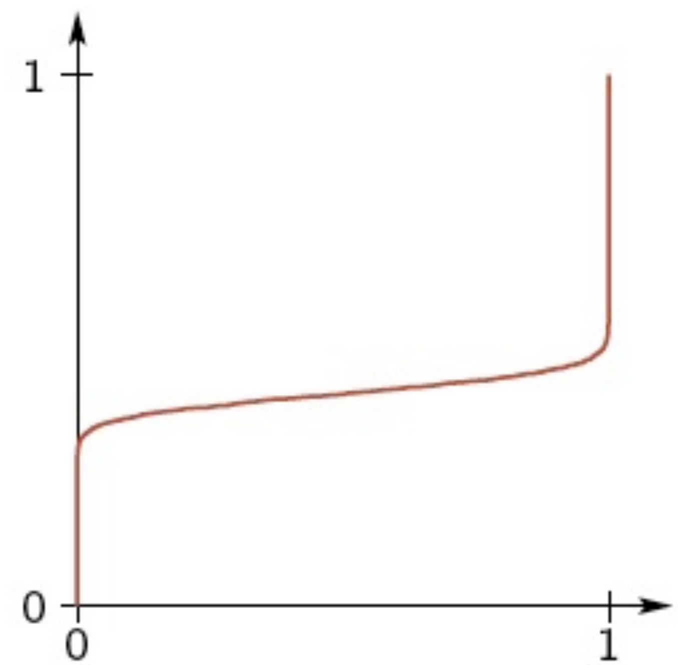


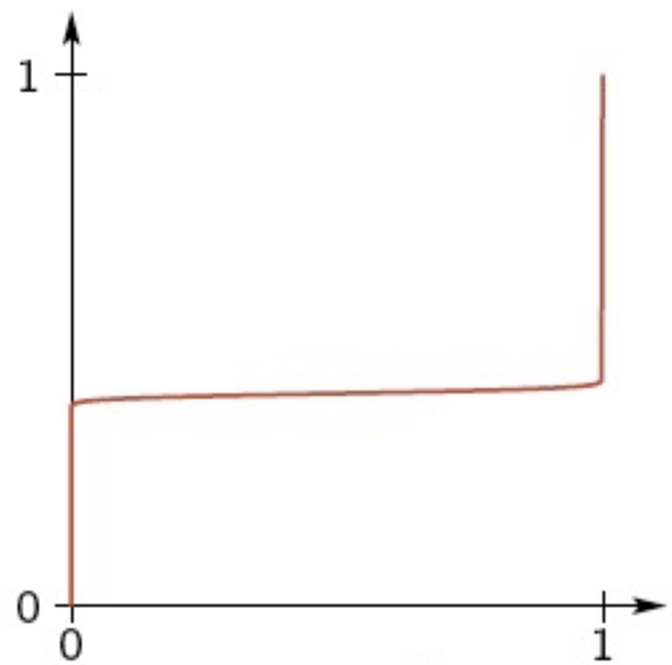
$N = 2^8$

Channel Polarization



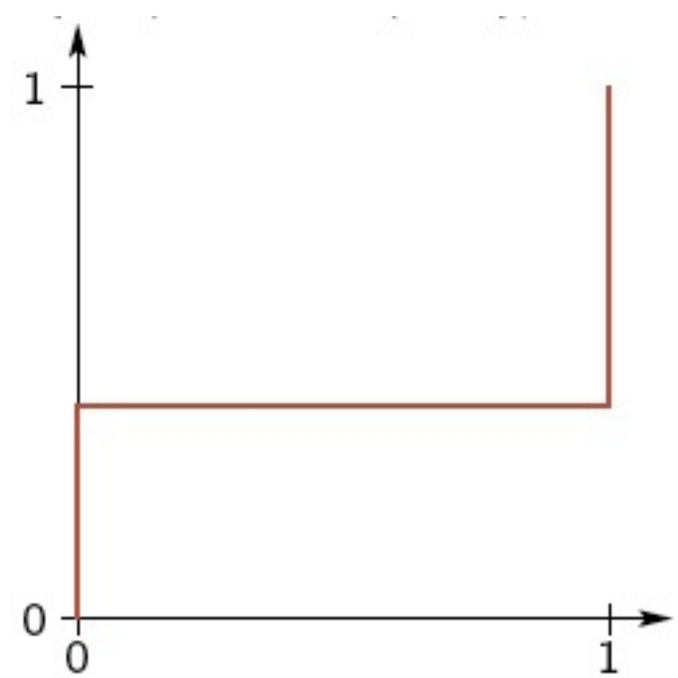
$N = 2^{12}$

Channel Polarization

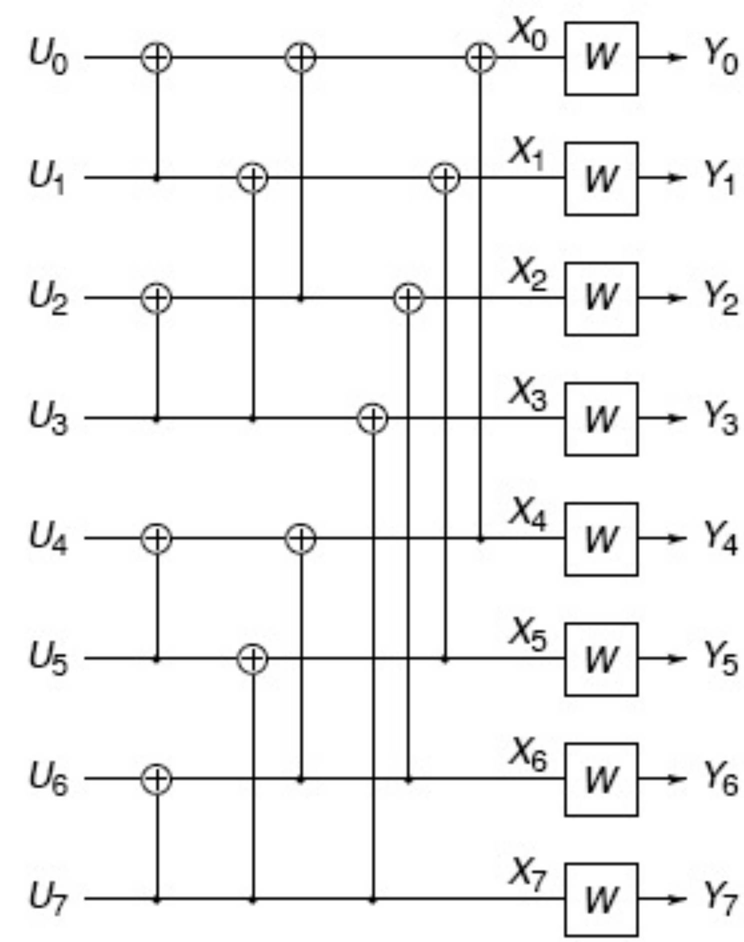


$N = 2^{20}$

Channel Polarization



Successive Decoding



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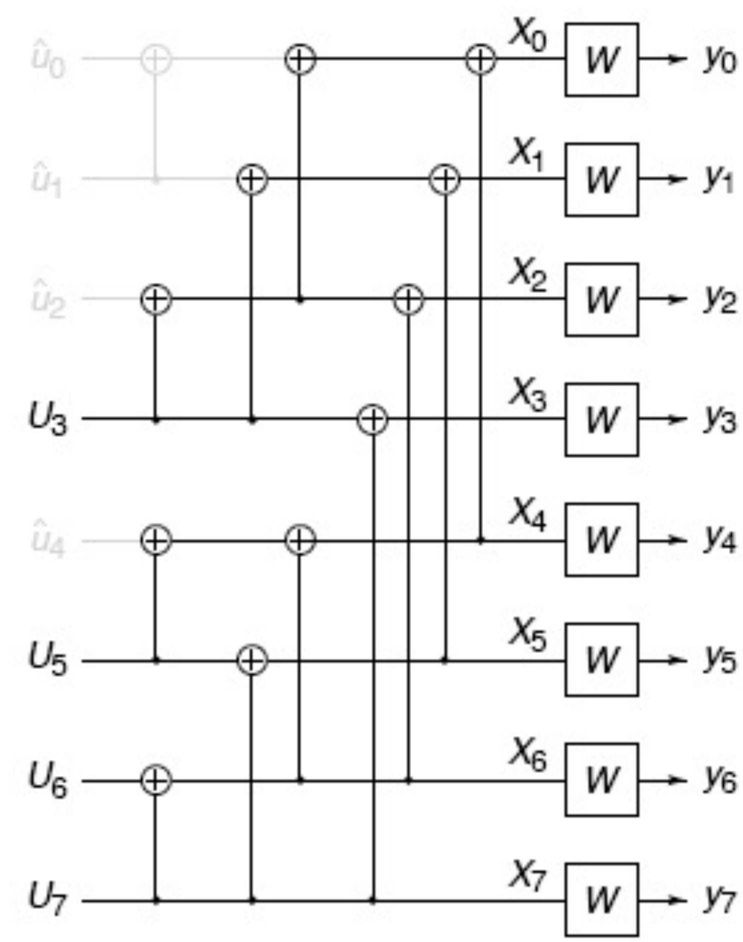

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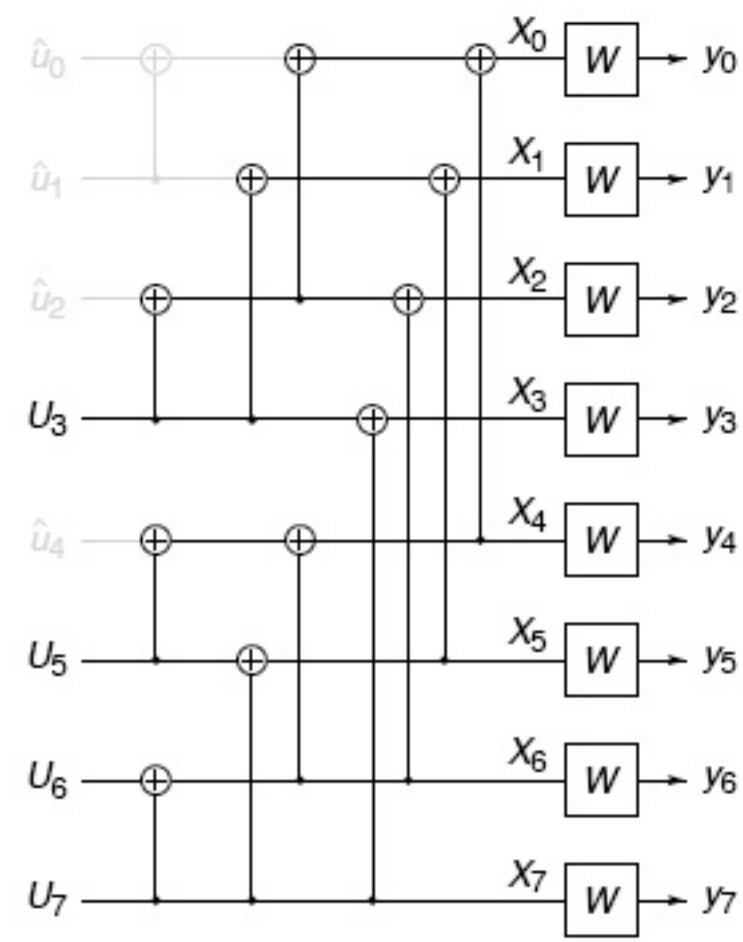

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if $i \in F$, $\hat{u}_i = 0$
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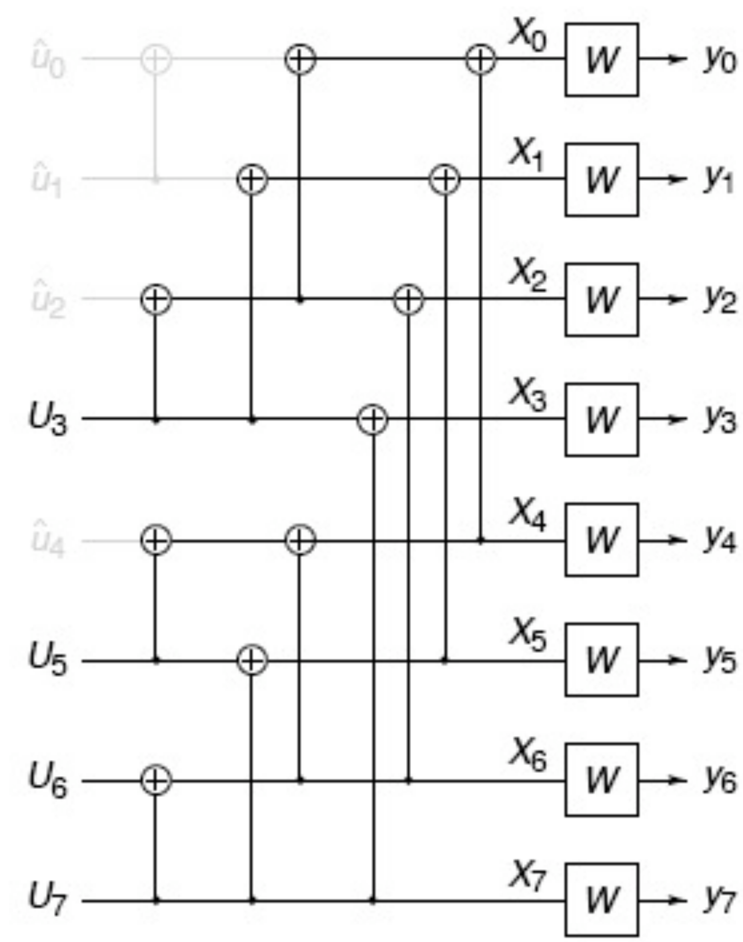

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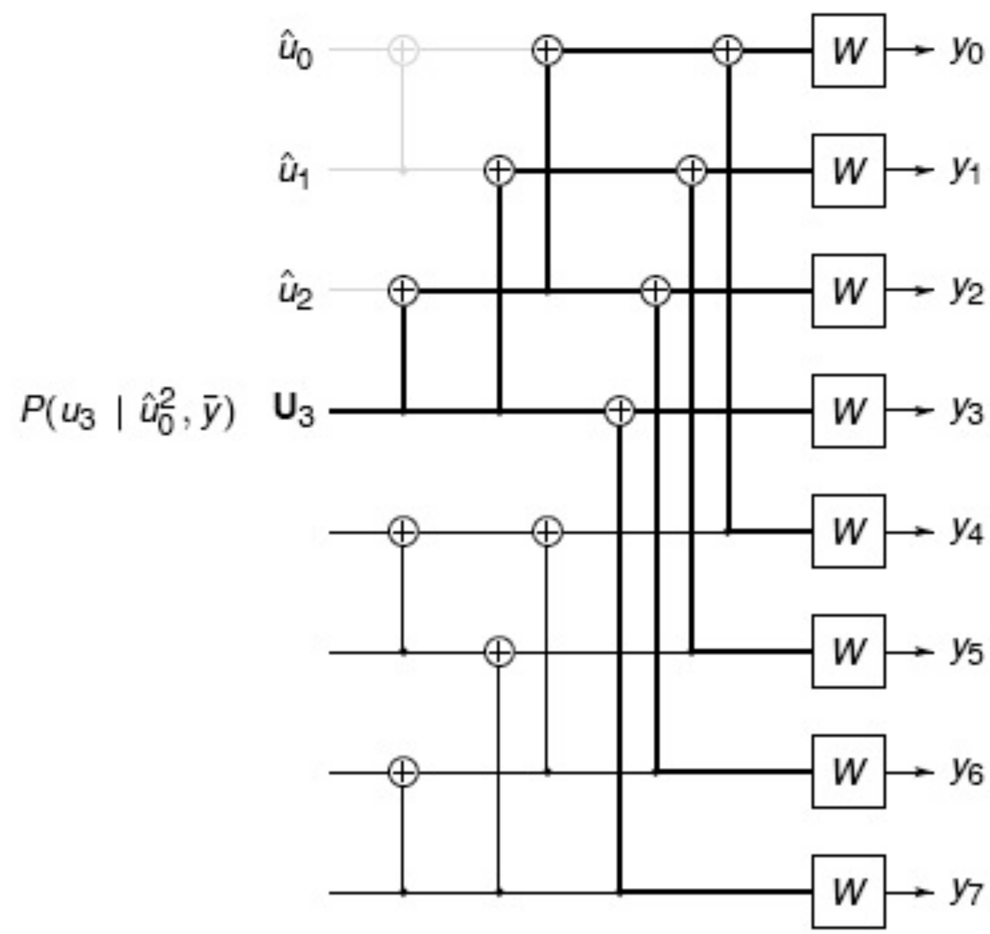

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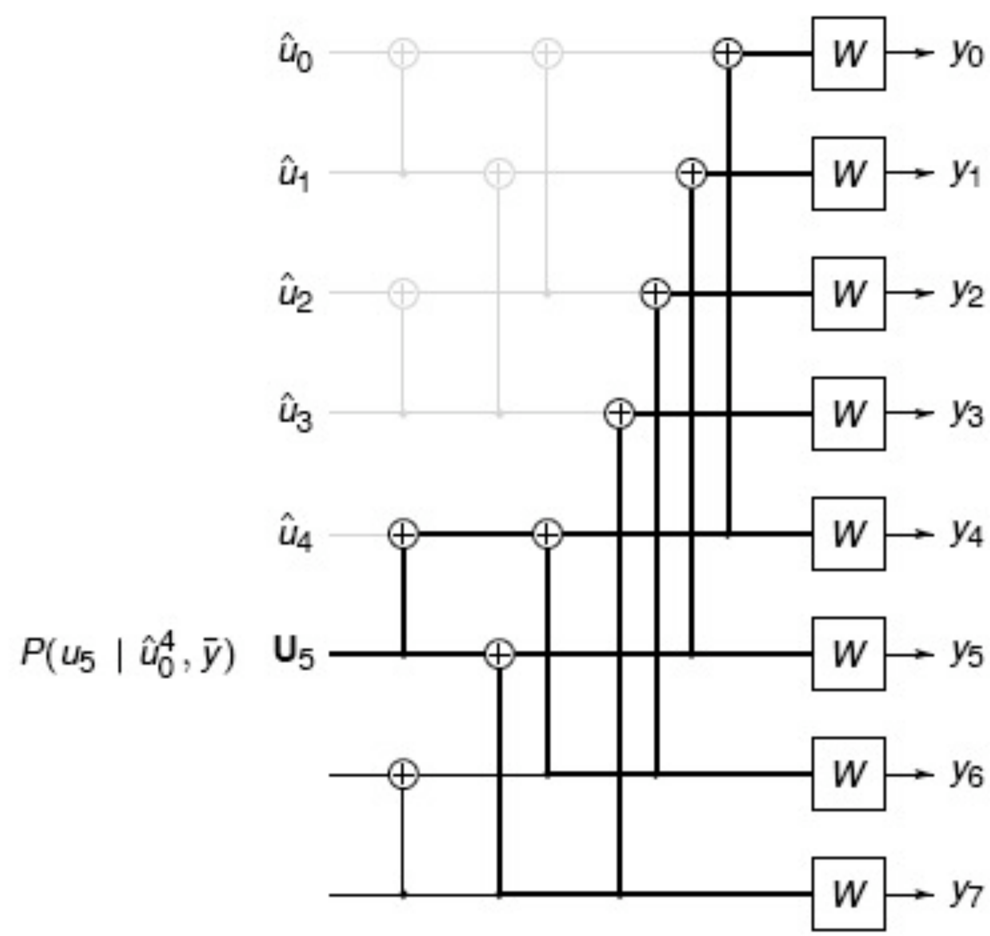

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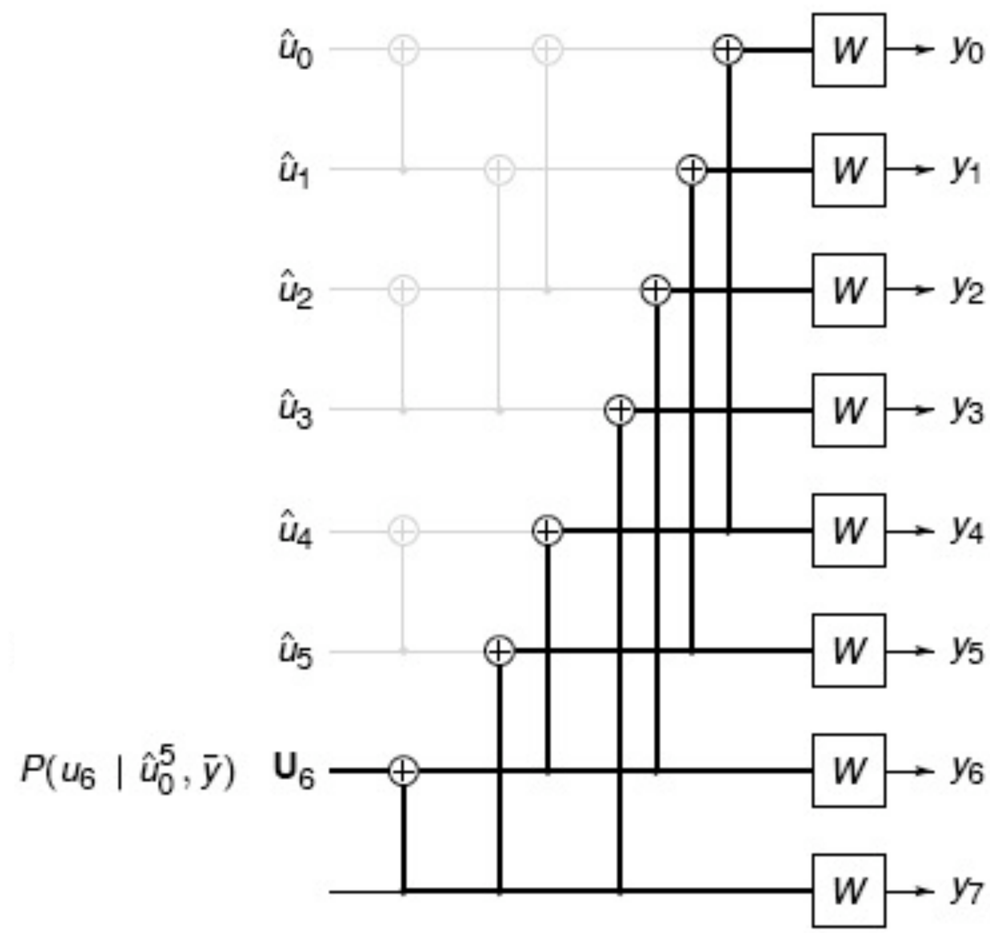

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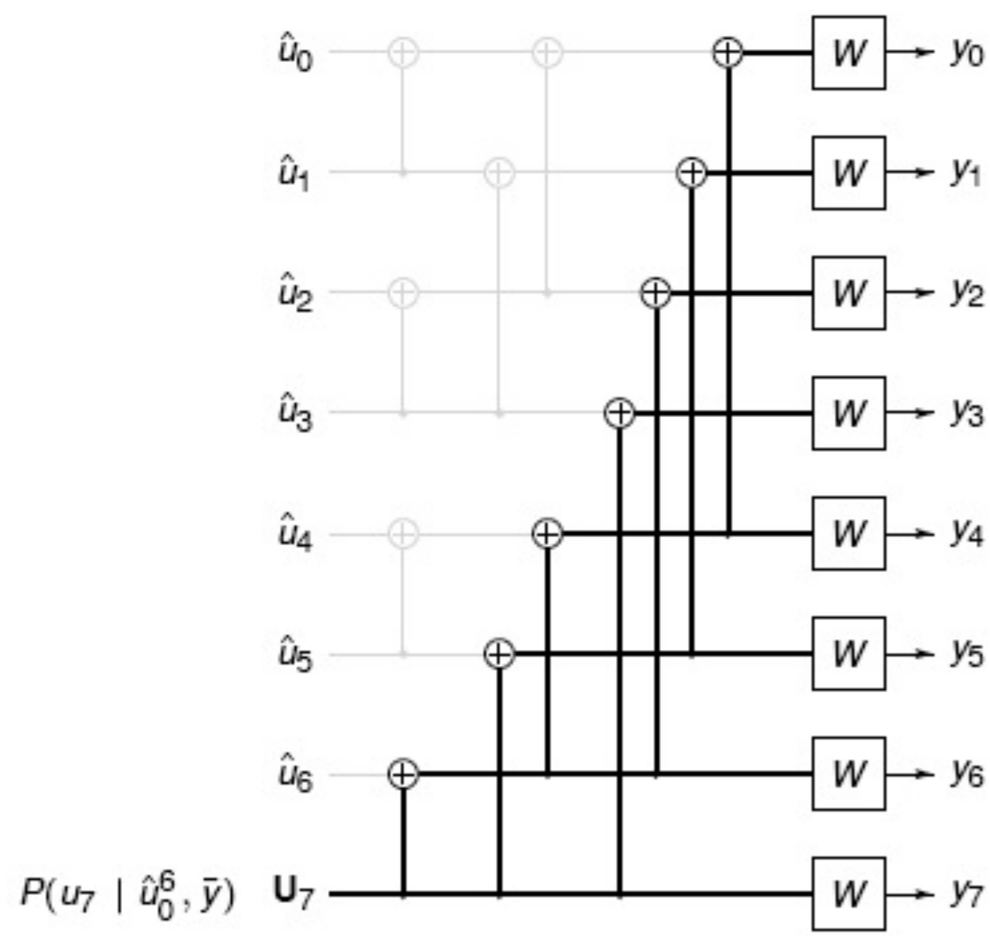

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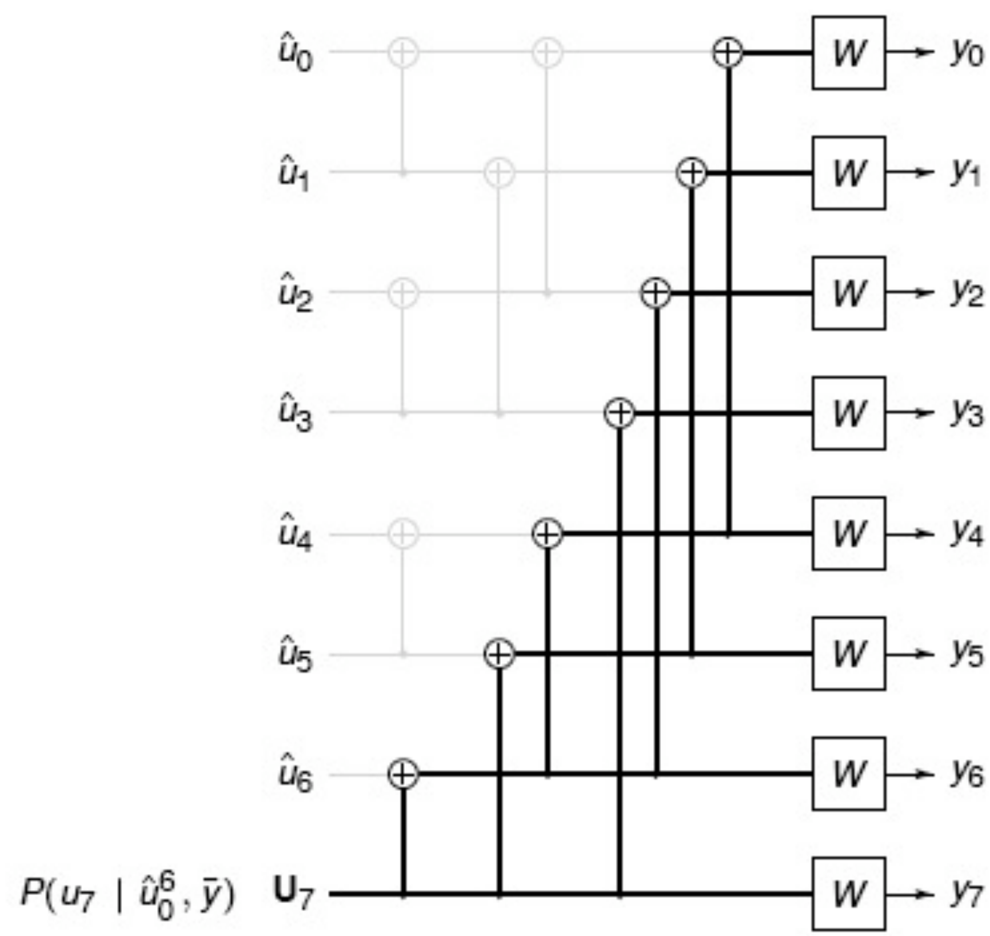
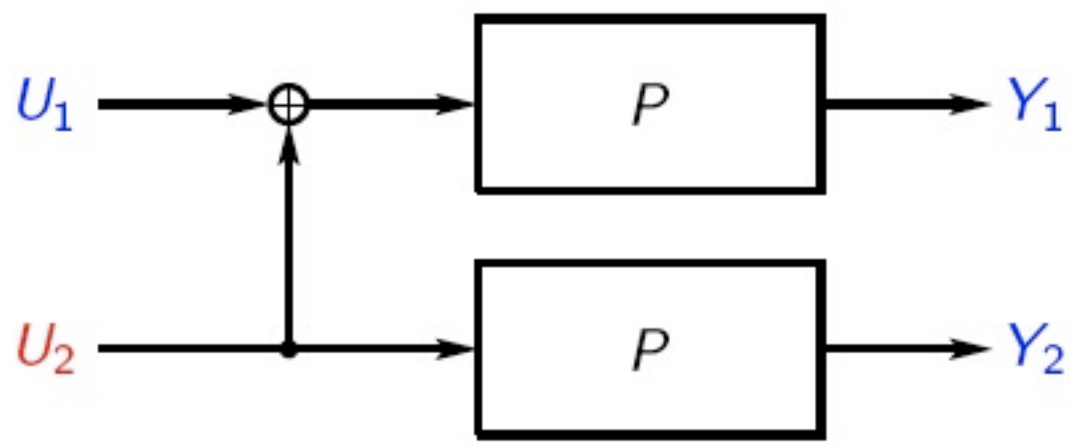
Complexity $O(N \log N)$



More on Polarization



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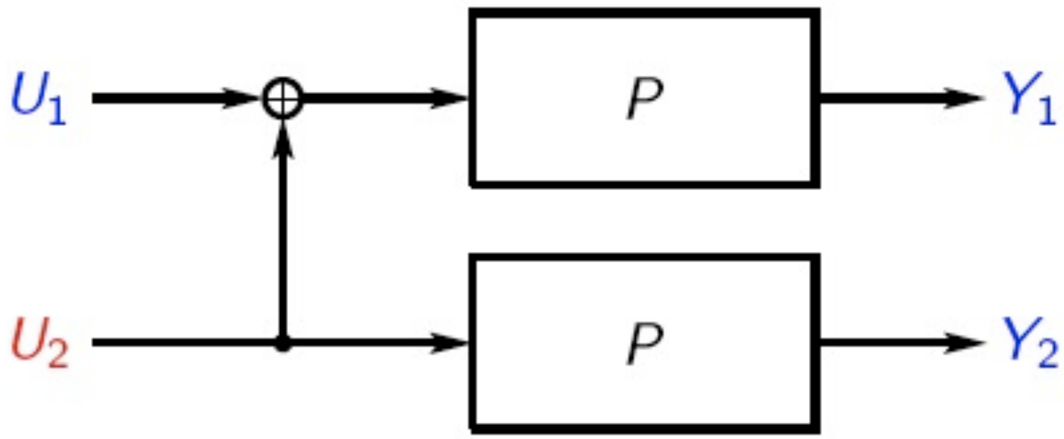
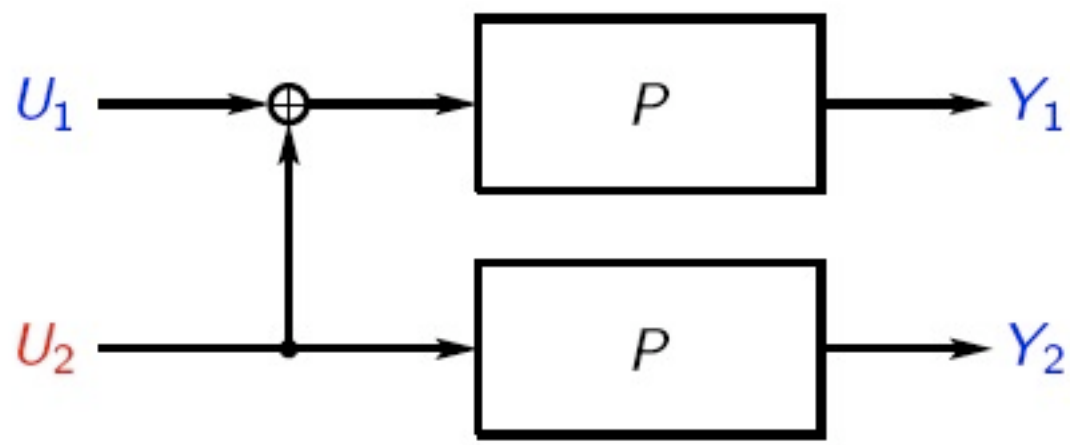


$$P^-(y_1 y_2 | u_1) = \sum_{u_2} \tfrac{1}{2} P(y_1 | u_1 + u_2) P(y_2 | u_2)$$

$$P^+(y_1 y_2 u_1 | u_2) = \tfrac{1}{2} P(y_1 | u_1 + u_2) P(y_2 | u_2)$$

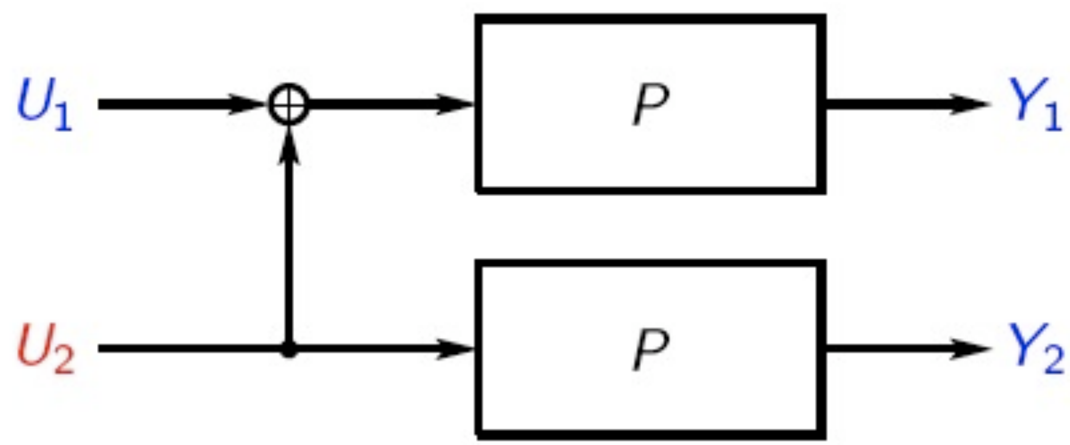
More on Polarization



$$I(P^-) = I(U_1; Y_1 Y_2)$$

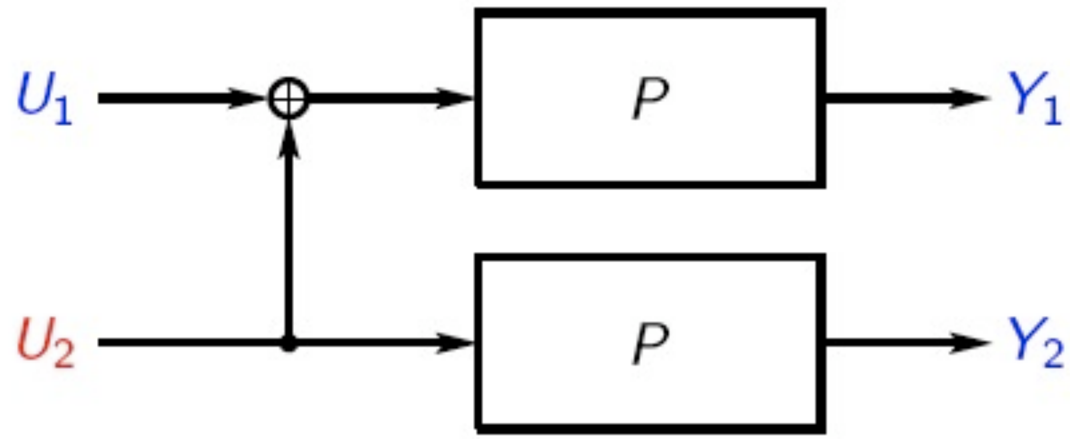
$$I(P^+) = I(U_2; Y_1 Y_2 U_1)$$

More on Polarization



$$
\begin{aligned}
I(P^-) + I(P^+) &= I(U_1 U_2; Y_1 Y_2) \\
&= I(X_1; Y_1) + I(X_2; Y_2) \\
&= 2I(P),
\end{aligned}
$$

More on Polarization



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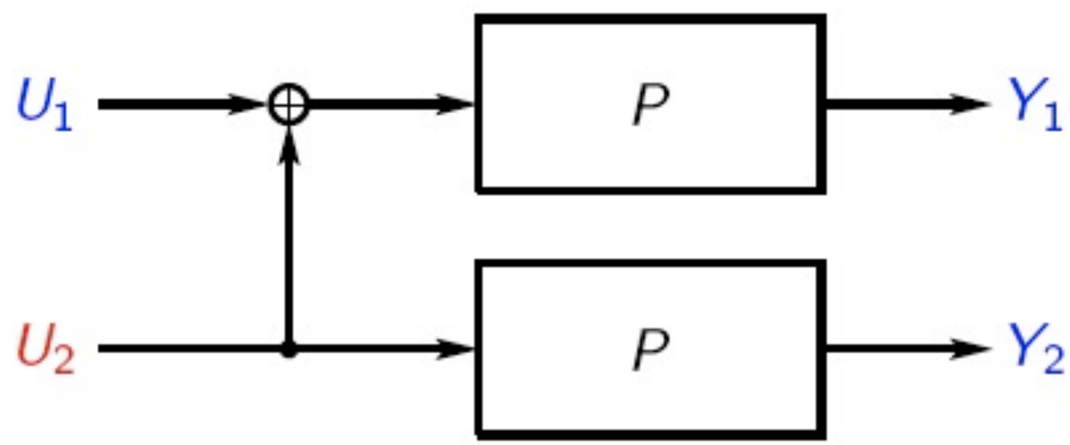

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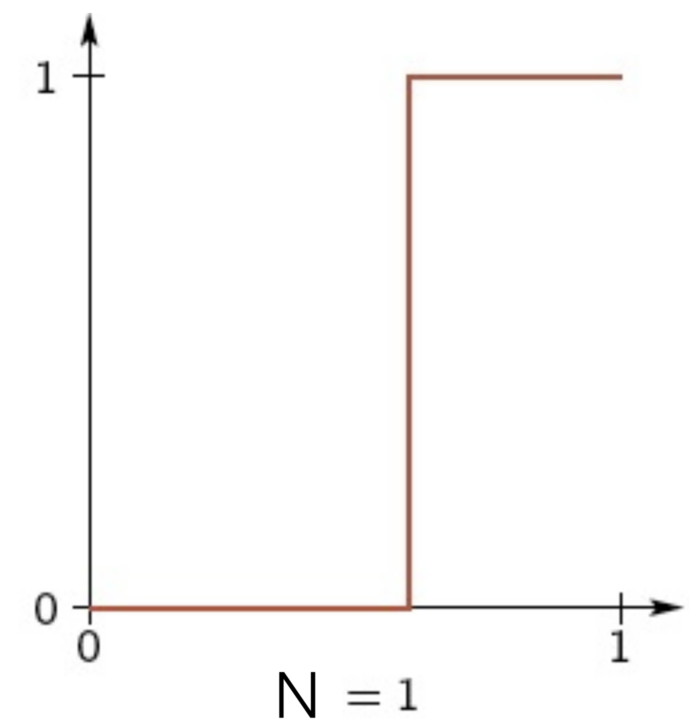

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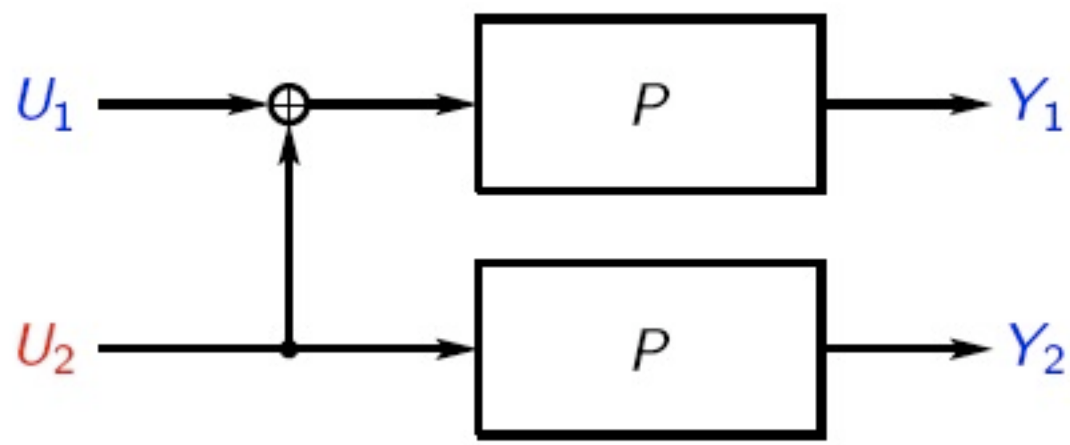

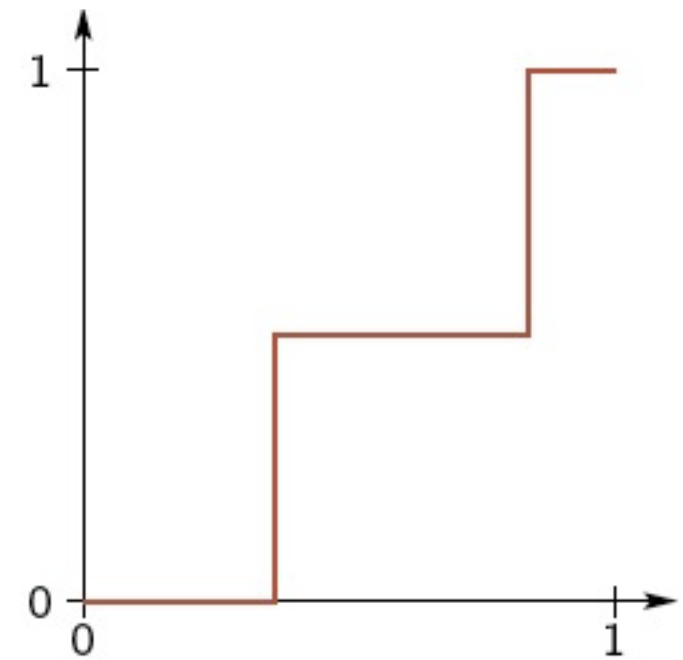

N $= 2$

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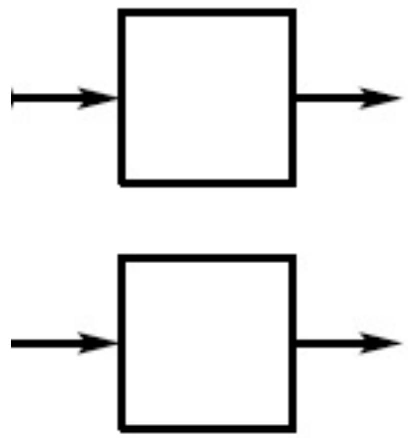

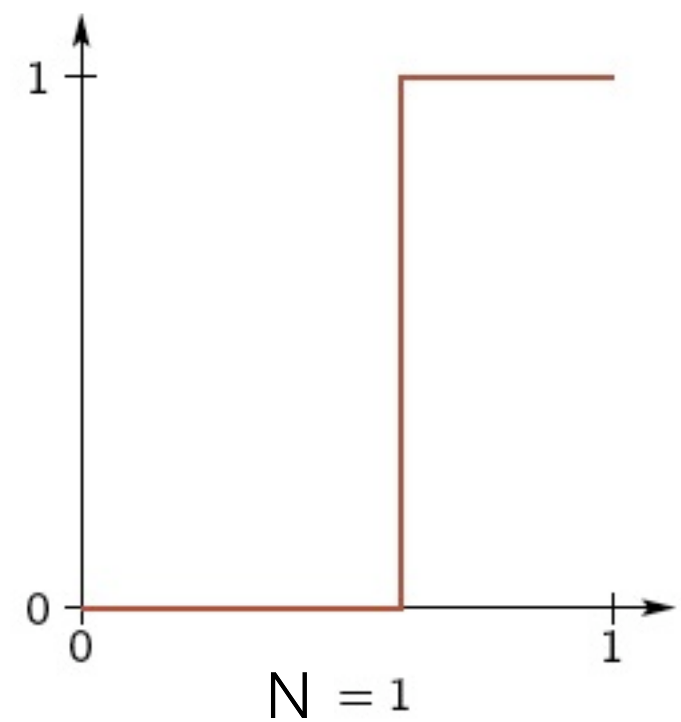

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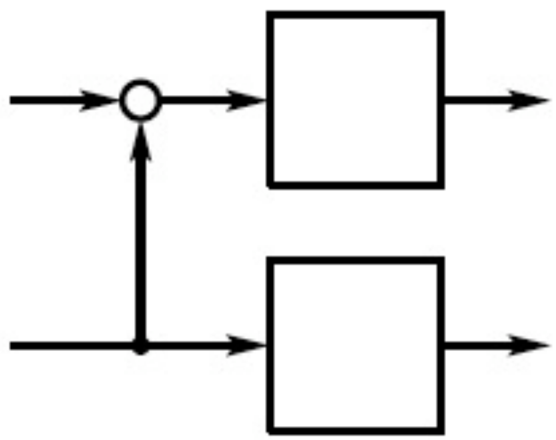

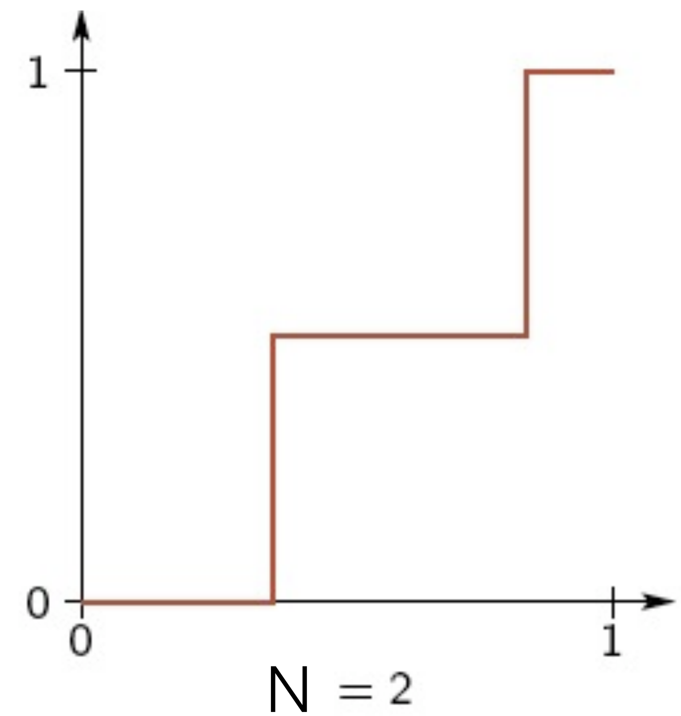

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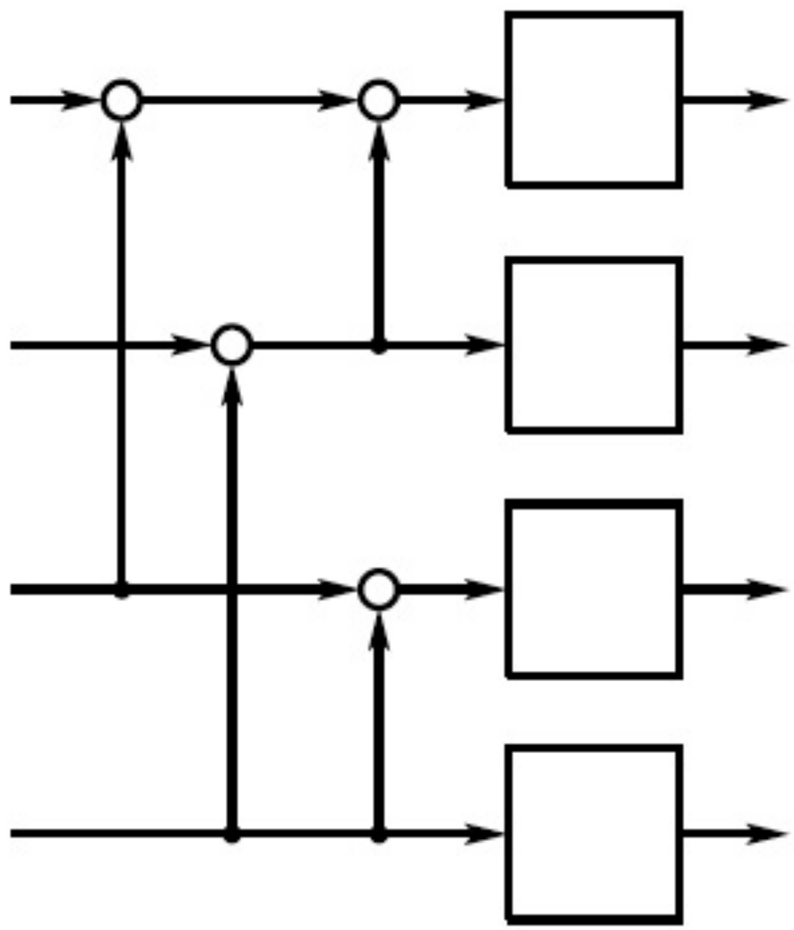

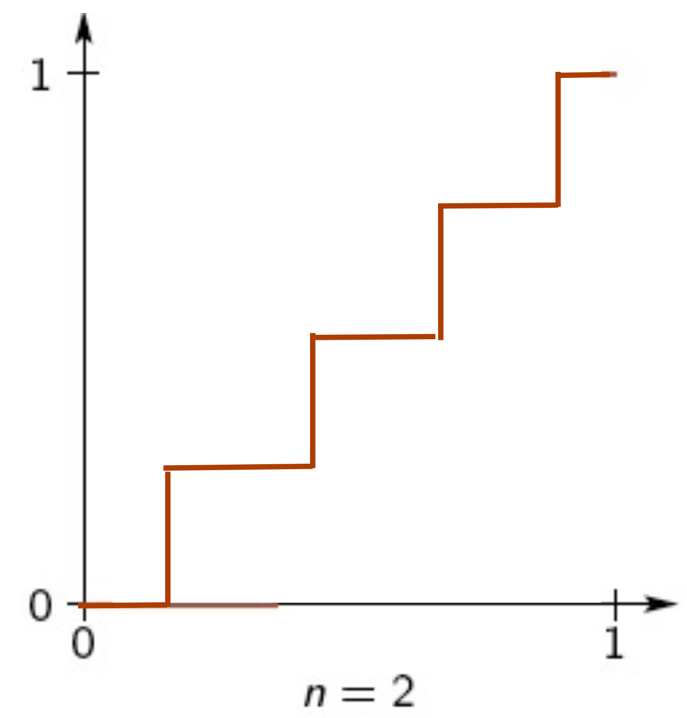

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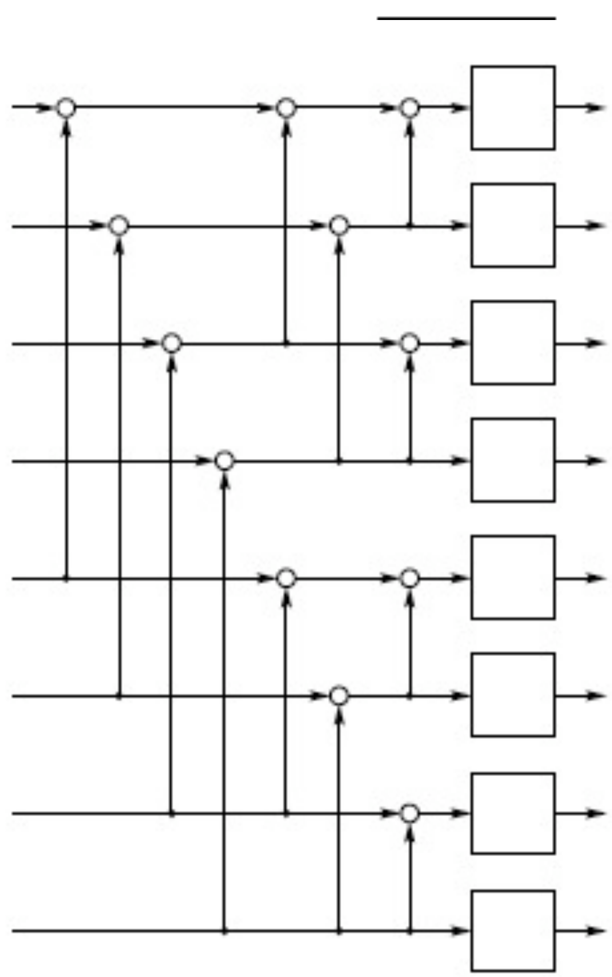

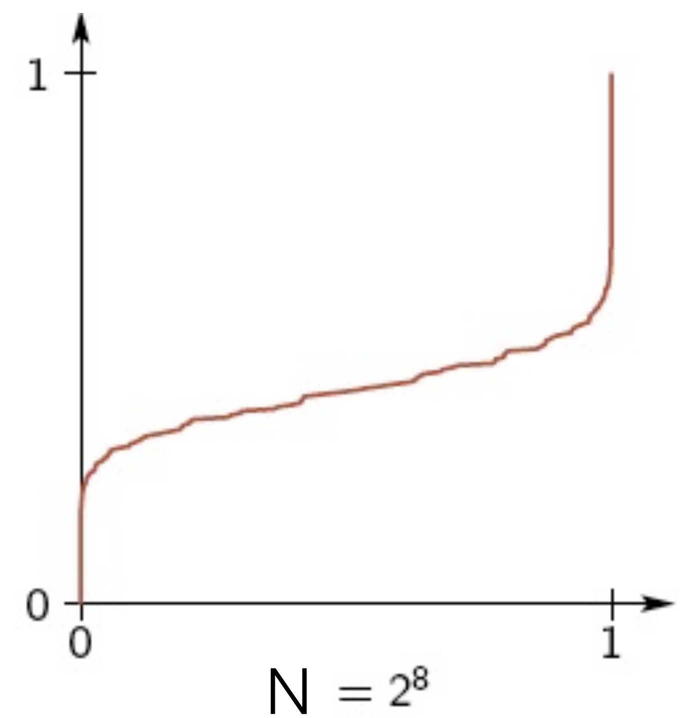

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Define $I_n = I(W^{B_1, B_2, \ldots, B_n})$

Study the distribution of I_n

Properties of I_n

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$I_0 = I(W)$ is a constant

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Further, $E[I_{n+1} \mid B_1, B_2, \ldots, B_n] = (I(P^+) + I(P^-))/2 = I(P)$, so $\{I_n\}$ is a (bounded) martingale

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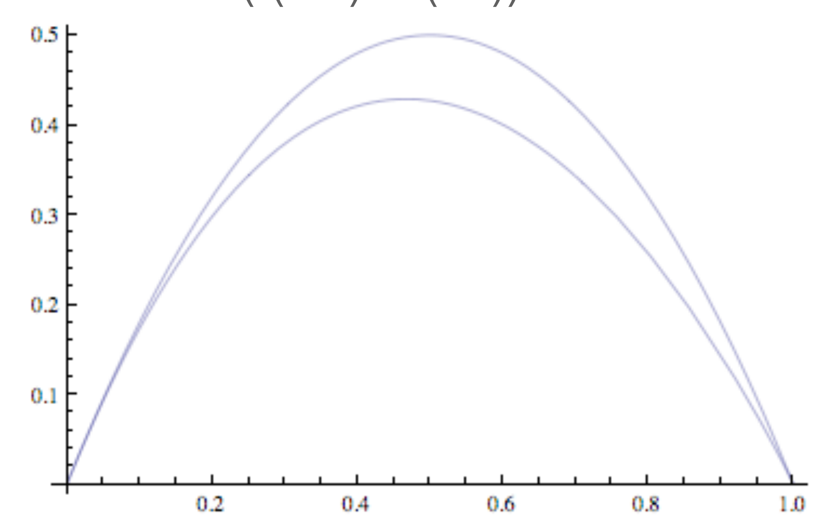

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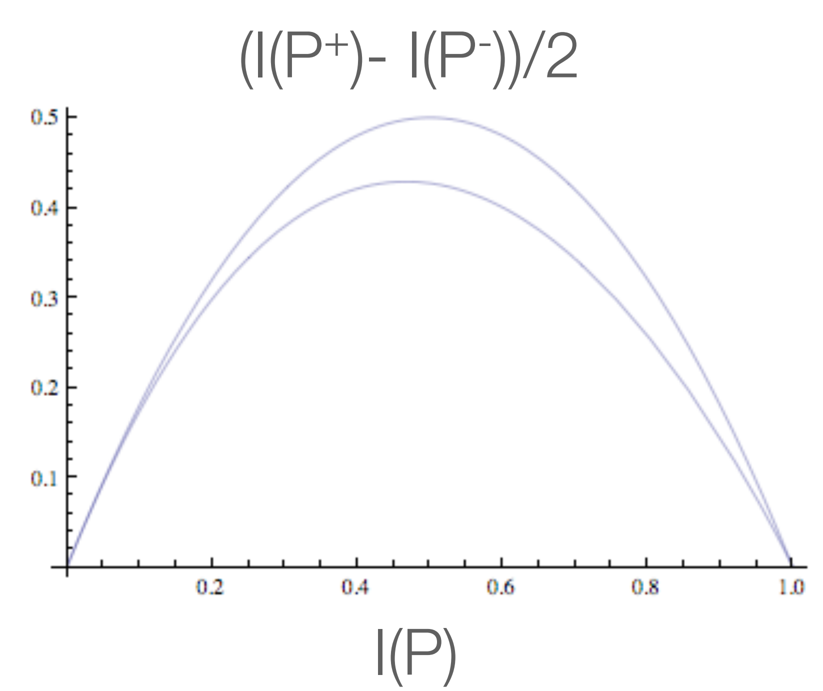
we conclude that I_∞ takes values only in $\{0, 1\}$



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Polar Codes Based on Larger Matrices

In [Arıkan08] generator matrix is constructed from the rows of $G_2^{\otimes m}$, where

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Consider codes constructed from $G^{\otimes m}$, where G is an $\ell \times \ell$ matrix. Blocklength $N = \ell^m$, $\bar{x} = \bar{u}G^{\otimes m}$

Characterization of Exponent

Theorem (Korada,Şaşoğlu,Urbanke)

For any $R < I(W)$, and $\ell \times \ell$ matrix G,

$$P_B(N) \approx 2^{-(N)^{\mathtt{E}_c(G)}},$$

where $\mathtt{E}_c(G) = \frac{1}{\ell}\sum_{i=1}^{\ell} \log_\ell d_i$.

$$G = \begin{bmatrix} g_1 \\ \vdots \\ g_i \\ g_{i+1} \\ \vdots \\ g_\ell \end{bmatrix} \qquad d_i = \mathtt{dmin}(g_i, \langle g_{i+1}, \ldots, g_\ell \rangle)$$

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$$\mathtt{E}_c(G) = \frac{1}{3}(\log_3 1 + \log_3 2 + \log_3 2) = 0.42062$$

Results

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$$\mathtt{E}_\ell = \max_{G \in \{0,1\}^{\ell \times \ell}} \mathtt{E}_c(G)$$

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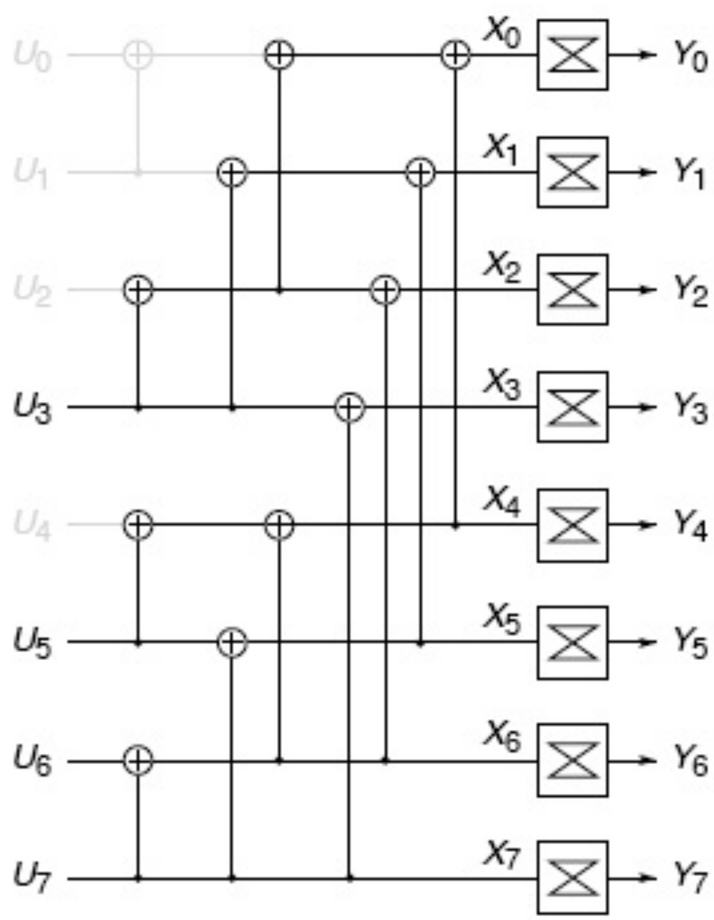
Source Coding



Y-binary symmetric source

$\mathtt{d}(0,1) = 1$, $\mathtt{d}(0,0) = \mathtt{d}(1,1) = 0$

$\bar{Y}$- source word, $\bar{X}$-reconstruction wor

$\frac{1}{N}\mathbb{E}[\mathtt{d}(\bar{X}, \bar{Y})] \leq D$

$R > 1 - h_2(D)$

Source Coding

- trellis based quantization [Viterbi and Omura], constraint length to infinity
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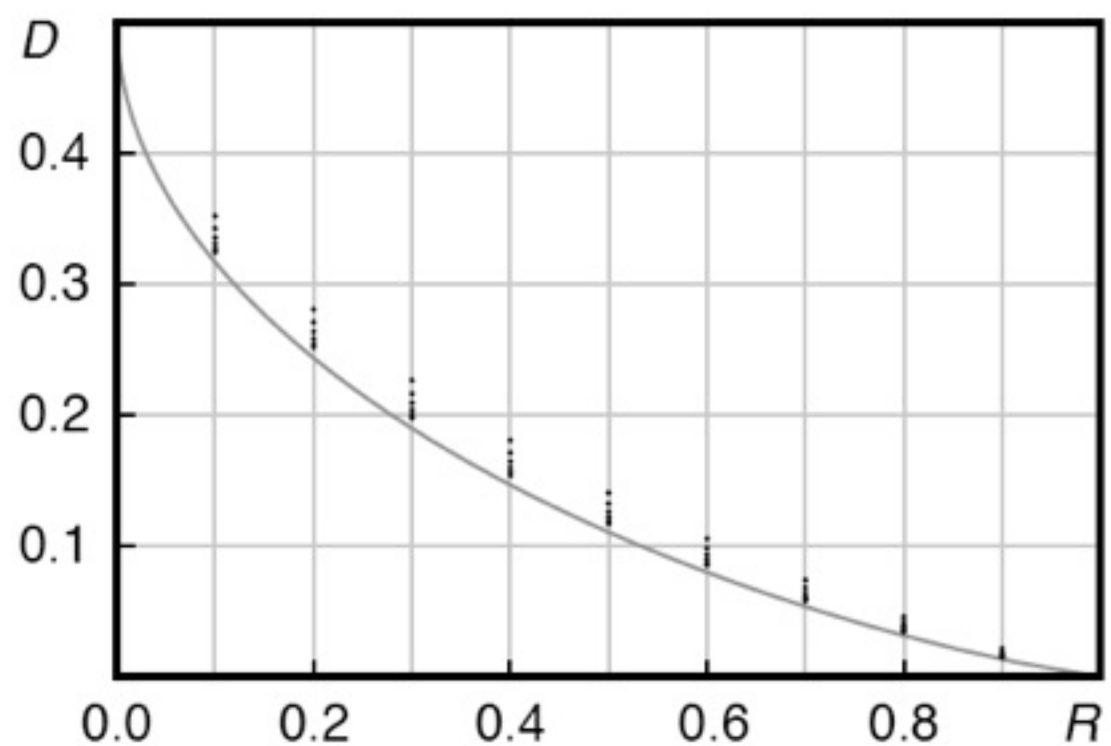

Length = $2^9, 2^{11}, \ldots, 2^{19}$

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scaling	

Summary

+ completely new paradigm of coding

+ provably achieves capacity

+ low complexity

+ many applications

- currently only competitive for VERY large N

Wyner-Ziv and Gelfand-Pinsker

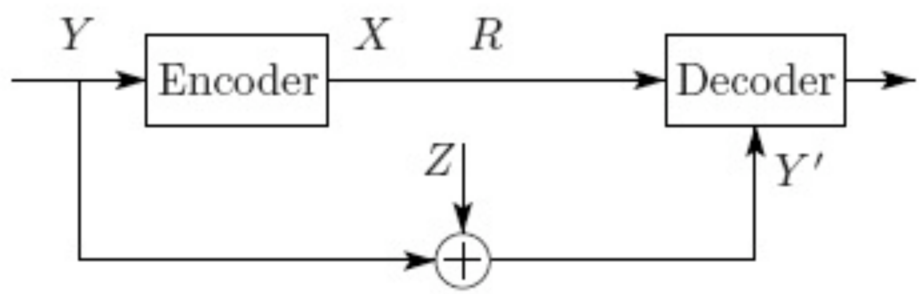

Figure 4.1: The Wyner-Ziv problem. The task of the decoder is to reconstruct the source Y to within a distortion D given (Y', X).

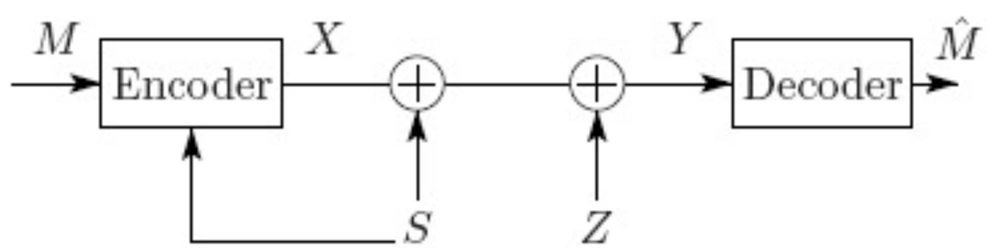

Figure 4.4: The Gelfand-Pinsker problem. The state S is known to the encoder a-causally but not known to the decoder. The transmission at the encoder is constrained to have only a fraction D of ones on average, i.e., $\mathbb{E}[X] \leq D$.